Homology Theory Cheat Sheat

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1 Introduction

These notes are intended to be a rapid primer on concepts and results in homology theory that are essential for understanding manifold topology and, more specifically, symplectic topology. Homology theory is the source of many of the most useful and easily computed manifold invariants, so it is very important to have at least a working knowledge of it before beginning an in depth look at manifolds.

The essential idea of homology theory is, in broad strokes, the following. You want a way to tell two spaces X and Y apart. The approach of algebraic topology to this problem is to devise a method or procedure $X \rightsquigarrow G(X)$ for taking any space X and acquiring an associated algebraic object $G(X)$ (say, a group, ring, vectorspace, module, algebra, etc, etc), so that one may compare $G(X)$ and $G(Y)$ instead. Ideally this method is amenable to computation, so that you can actually calculate $G(X)$ completely given some reasonable data about X. It is also usually crucial for this process $X \rightsquigarrow G(X)$ to have the property that, given a map $f : X \to Y$, you can get a group map $G(f) : G(X) \to G(Y)$ (or perhaps in the opposite direction). This way, one can do nice manipulations with $G(X)$ and $G(Y)$ to divine things about the relationship between X and Y.

Homology and cohomology theories are essentially procedures $X \rightsquigarrow G(X)$ in this vein, which satisfy a series of very nice axioms that particularly simplify things and make computation easier. It is difficult to overstate their usefulness and centrality to modern differential topology.

In Section 2, we will give a definition for a homology theory via the Eilenberg-Steenrod axioms (ignoring, for our purposes, extraordinary homology theories such as K-theory and embedded cobordism groups that may not quite fit these criteria). In Section 3, we will outline the typical method of constructing a homology theory and the results in homological algebra that make this possible. In Section 4, we will mention lots of examples of homology theories and discuss in some detail the two most important examples for ur, cellular homology and de Rham cohomology. In Section 5, we will mention some important additional properties of homology on manifolds: fundamental classes and Poincare duality. In Section 6 we will perform some example computations.

2 Eilenberg-Steenrod Axioms

We must start by recalling the definition of the category of pairs of topological spaces, which we will call **Top2**. We will define homology groups as a collection of homotopy invariants for such pairs. The objects of this category are pairs (X, A) of topological spaces with $A \subset X$. The morphisms are continuous maps $\phi : X \to Y$ between the first terms of pairs (X, A) and (Y, B) such that $\phi(A) \subset B$. Note that the category of topological spaces is a sub-category, consisting of the objects $(X, \emptyset)$ and all the morphisms between any two such pairs. We will often abbreviate the pair $(X, \emptyset)$ simply as X.

Definition 2.1. Two morphisms $f, g : (X, A) \to (Y, B)$ are **homotopic** if there exists a continuous map $f_t : [0, 1] \times X \to Y$ such that $f_t(A) \subset B$ for all t, $f_0 = f$ and $f_1 = g$ □

We can similarly define the category of pairs of compact manifolds **Man2** and pairs of CW-complexes **CW2**. The definitions of morphisms and homotopies there are exactly analogous.

Definition 2.2. (Eilenberg-Steenrod Axioms) Before we start getting into constructions and examples, we're going to provide a set of axioms

A **homology theory** is a collection of functors $H_i : \mathbf{Top2} \to \mathbf{Ab}$ (one for each non-negative integer i) along with a natural transformation $\partial : H_i(X, A) \to H_{i-1}(A)$ (called the **connecting homomorphism**) satisfying:

1. **Homotopy** If $f, g : (X, A) \to (Y, B)$ are two homotopic maps then the induced maps $f_*, g_* : H_i(X, A) \to H_i(Y, B)$ are the same.[1]
2. **Excision** If $U \subset X$ satisfies $\bar{U} \subset \text{int}(A) \subset X$ then the map $i_* : H^i(X - U, A - U) \to H^i(X, A)$ induced by the inclusion $i : (X - U, X - U) \to (X, A)$ is an isomorphism.
3. **Dimension** If P is the 1 point space, then $H_i(P) = 0$ for all $i > 0$.
4. **Additivity** If $X = \sqcup_\alpha X_\alpha$ then $H_i(X) \simeq \oplus H_i(X_\alpha)$.
5. **Exactness** Each pair (X, A) induces a long exact sequence in homology, via the string of inclusions $(A, \emptyset) \xrightarrow{i} (X, \emptyset) \xrightarrow{j} (X, A)$:

$$\ldots \xrightarrow{j_*} H_{i+1}(X, A) \xrightarrow{\partial} H_i(A) \xrightarrow{i_*} H_i(X) \xrightarrow{j_*} H_i(X, A) \xrightarrow{\partial} H_{i-1}(A) \xrightarrow{i_*} \ldots$$

   Note that the maps $H_i(X, A) \to H_{i+1}(X, A)$ are given by ∂.

A **cohomology theory** is a collection of contravariant functors H^i : **Top2** $\to$ Ab along with a natural transformation $\partial : H^i(X, A) \to H^{i+1}(A)$ (also called the **connecting homomorphism**) satisfying (1)-(5) with all maps f_* replaced with f^* [2] and all the arrows reversed.

We can also use the same definition for functors over **CW2** and **Man2**. The above axioms are very restrictive. In fact, one can prove the following result.

Theorem 2.1. If H^i and $\tilde{H}^i$ are two homology (resp. cohomology) theories satisfying the Eilenberg-Steenrod axioms for the category **Man2**. Furthermore suppose that $H^0(P) \simeq \tilde{H}^0(P)$. Then $H^i(X, A) \simeq H^i(X, A)$.

For a manifold, then, any homology theory satisfying the Eilenberg-Steenrod axioms with $H^0(P) \simeq R$ can be denoted as $H^i(X; R)$ (without specifying its specific construction) and referred to somwhat unambiguously as **manifold homology** or **ordinary homology**.

Remark 2.1. We will see later that, as disappointing as this result sounds, it is in fact very useful.

[1] It is convention to denote the morphism $H_i(f) : H_i(X, A) \to H_i(Y, B)$ induced by a morphism $f : (X, A) \to (Y, B)$ by f_*, suppressing the dependence on i and alluding to the notation used for pushforward of vector-fields in manifold theory.

[2] As with homology, it is conventional to denote $H^i(f) : H^i(Y) \to H^i(X)$ by f^*.

3 Constructing A Homology Theory

The above axioms don't tell you how to find homology theories or how to calculate $H_i(X)$. In applications, homology is generally computed using instructions on how to define a chain complex $C_i(X)$ from X, then computing the homology of $C_i(X)$. First let us define chain complexes and their homology.

Definition 3.1. A **chain complex** (C_*, ∂) of R-modules is a sequence of free R-modules C_i, called the **chain groups**, along with a sequence of maps $\partial_i : C_i \to C_{i-1}$, called the **chain differentials**, such that $\partial_i \circ \partial_{i+1} = 0$. Usually the subscript i is suppressed in the notation for ∂_i, and every ∂_i is simply called ∂. The chain groups now fit into a sequence:

$$\ldots \xrightarrow{\partial} C_i \xrightarrow{\partial} C_{i-1} \xrightarrow{\partial} \cdots \to C_1 \xrightarrow{\partial} C_0 \xrightarrow{\partial} 0$$

A **co-chain complex** (C^*, δ) of R-modulules is a sequence of free R-modules C^i, called the **co-chain** groups, along with a sequence of maps $\delta_i : C^i \to C^{i+1}$, called the **chain co-differentials** (or just the **chain differentials**), such that $\delta_{i+1} \circ \delta_i = 0$[3] Again, the subscript is usually suppressed. The cochain groups fit into their own sequence, with the arrows going in reverse. □

Definition 3.2. The **homology** $H_*(C_*, \partial)$ of a chain complex is defined as the sequence of groups $H^i(C^*, \partial)$ defined as:

$$H_i(C_*, \partial) := \ker(\partial_i)/\mathrm{im}(\partial_{i+1}) = \frac{\ker(\partial : C_i \to C_{i-1})}{\mathrm{im}(\partial : C_{i+1} \to C_i)}$$

Where $A/B = \frac{A}{B}$ here means quotient as R-modules. The **cohomology** $H^*(C^*, \delta)$ of a co-chain complex is defined the same way:

$$H^i(C^*, \delta) := \ker(\delta_i)/\mathrm{im}(\delta_{i-1})$$

□

Definition 3.3. A **chain map** $F : C_* \to D_*$ between two chain complexes (C_*, ∂^C) and (D_*, ∂^D) is a set of linear maps $F_i : C_i \to D_i$ such that $\partial_i^D F_i = F_i \partial_i^C$ (this is usually written as $\partial F = F\partial$). Likewise, a **co-chain map** $F : C^* \to D^*$ between two cochain complexes (C^*, δ^C) and (D^*, δ^D) is a set of linear maps $F_i : C^i \to D^i$ such that $\delta_i^D F_i = F_i \delta_i^C$ (again, usually written as $F\delta = \delta F$).

Exercise 3.1. Show that a chain map $F : C_* \to D_*$ induces well-defined R-module maps $[F]_i : H_i(C_*, \partial) \to H_i(D^*, \partial)$ by using the quotient definition of homology.

Remark 3.1. If (C_*, ∂) is a chain complex, we may form the **dual co-chain complex** by defining $C^i = \mathrm{Hom}(C_i, R)$ (where R is the base ring) and defining $\delta : C^i \to C^{i+1}$ by the identity:

$$\langle \delta\alpha, c \rangle = \langle \alpha, \partial c \rangle$$

for any $\alpha \in C^i$ and $c \in C_{i+1}$.

[3]It is convention to use subscript indices (i.e H_i, C_i) for homology groups and chain groups and super-script (i.e H^i, C^i) for cohomology groups and cochain groups. Furthermore, it is convention to use ∂ for chain differentials and δ for cochain differentials.

Exercise 3.2. Verify that the dual co-chain complex defined above is in fact a co-chain complex. Show that any chain map $F : C_* \to D_*$ induces a dual co-chain map $F^* : C^* \to D^*$.

Exercise 3.3. Let (C_*, ∂) be a chain complex and (C^*, δ) be its dual. Show that there is a well-defined map:

$$\phi : H^i(C^*, \delta) \to \mathrm{Hom}(H_i(C_*, \partial), R)$$

given by:

$$\langle \phi([\alpha]), [a] \rangle = \langle \alpha, a \rangle$$

Here $[\alpha] \in H^i(C^*, \delta)$ and $[a] \in H_i(C_*, \partial)$. $\alpha \in C^i$ and $a \in C_i$ are representatives of $[\alpha]$ and $[a]$ respectively. We will usually denote $\langle \phi([\alpha]), [a] \rangle$ as simply $\langle [\alpha], [a] \rangle$.

A homology theory H_i will typically be defined by some instructions or rules dictating how to take a space X and acquire a chain complex $(C_*(X), \partial)$. We will denote such instructions abstractly by this silly squiggly arrow $X \rightsquigarrow (C_i(X), \partial)$, just so we have some way to refer to them. Then for a given space X, the homology groups H_i for X are defined as $H_i(X) = H_i(C_*(X), \partial)$.

Remark 3.2. The groups $C_i(X)$ need **not** be unique for a given X: the instructions $X \rightsquigarrow (C_i(X), \partial)$ may dictate that some number of arbitrary choices must be made while creating $C_i(X)$, and different choices may result in different groups $C_i(X)$. However, for a well-defined homology theory, two different chain complexes $C_*(X)$ and $C'_*(X)$ acquired via the instructions will produce naturally isomorphic homology, i.e $H_i(C_*(X), \partial) \simeq H_i(C_*(X), \partial)$, so that the choices are not reflected in the groups $H_i(X)$. □

Remark 3.3. It is **not** a guarantee that any construction like this will satisfy the axioms given in Definition 2.2. These axioms often have to be verified when a new homology theory is formulated. □

The takeaway of this discussion is the following. Whenever you are confronted with a new homology theory for manifolds, there are 3 questions that you should immediately ask:

1. **Chain Groups** How do you define the chain groups?
2. **Chain Differential** How do you define the chain differentials?
3. **Induced Maps** (If it isn't clear) Given $f : X \to Y$, how do you define f_*? This information is usually specified as a map on the chains $f_* : C^i(X) \to C^i(Y)$ which descends to a well-defined map $f_* : H^i(X) \to H^i(Y)$ on the quotient.

4 Examples Of Homology Theories

In this section, we will discuss a series of homology theories that are isomorphic to manifold homology $H_i(X; R)$ or manifold cohomology $H^i(X; R)$ for a closed manifold X. We will only define the

4.1 Singular Homology

Definition 4.1. The **standard n-simplex** Δ^n is the convex hull in $\mathbb{R}^n$ of the 0 vector e_0 and the unit vectors $e_1^n, \dots, e_n^n$ in $\mathbb{R}^n$. That is, it is all points in $\mathbb{R}^n$ of the form:

$$p = \sum_{i=0}^{n} x_i e_i^n \quad \text{with} \quad \sum_i x_i = 1$$

□

There are $n+1$ natural inclusions $\chi_k^n : \Delta^n \to \Delta^{n+1}$ corresponding to the $n+1$ faces of Δ^n. These are given by:

$$\chi_k^n(p) = \chi_k^n(\sum_{i=0}^{n} x_i e_i^n) = \sum_{i=0}^{k-1} x_i e_i^{n+1} + \sum_{i=k+1}^{n} x_{i-1} e_i^{n+1}$$

Definition 4.2. Given a Hausdorff space X, we denote by $\mathrm{Map}(\Delta^k, X)$ the set of continuous maps from $\Delta^k \to X$.

The singular homology groups $H_i(X;R) = H_{i,\mathrm{sing}}(X;R) := H_i(C^*(X;\mathbb{R}), \partial)$ are defined by:

1. **Chain Groups** The chain group $C_i(X;R)$ is the free R-module generated by the maps in $\mathrm{Map}(\Delta^k, X)$. That is, the chain group is the set of all formal sums:

   $$\phi = \sum_{\alpha=0}^{k} r_\alpha \phi_\alpha \quad \text{with} \quad \phi_\alpha \in \mathrm{Map}(\Delta^k, X) \quad \text{and} \quad r_\alpha \in R$$

   Note that these groups are huuuge, since the space of generators is massive.

2. **Chain Differential** The chain differential $\partial_i : C_i(X;R) \to C_{i-1}(X;R)$ is defined as so. Since $C_{i+1}(X,R)$ is generated by the maps $\phi \in \mathrm{Map}(\Delta^{k+1}, X)$, is suffices to specify $\partial_{i+1}(\phi)$ for any such ϕ.

   $$\partial_{i+1}(\phi) = \sum_{i=0}^{n} (-1)^i \phi \circ \chi_i$$

3. **Induced Maps** Given a continuous map $f : X \to Y$, we define $f_* : C_i(X;R) \to C_i(Y;R)$ as so. Again, it suffices to define it on the generators $\phi \in \mathrm{Map}(\Delta^i, X)$. There we define it by the obvious formula:

   $$f_*\phi = f \circ \phi \in \mathrm{Map}(\Delta^i, Y)$$

   Note that it is obvious from this definition that ∂ and f_* commute.

We can also define singular cohomology $H^i(X;R) = H^i_{\mathrm{sing}}(X;R) := H^i(C^*(X,R), \delta)$ as so. This uses the construction in Remark 3.1, but just for clarity we will discuss it again in this context.

1. **Co-chain Groups** The chain group $C^i(X;R) = \mathrm{Hom}(C^i(X;R), R)$ is the space of linear maps (module homomorphisms) to the base ring R.

2. **Co-chain Differential** The differential is defined as the dual map induced by ∂. More specifically, given a co-chain $\alpha \in C^i(X;R)$, we define the co-chain $\delta\alpha \in C^{i+1}(X;R)$ by its value on any $c \in C_{i+1}(X;R)$:

$$\langle \delta\alpha, c\rangle = \langle \alpha, \partial c\rangle \text{ for all } c \in C_{i+1}(X;R)$$

   Here $\langle \alpha, c\rangle$ denotes the dual pairing of α with c.

3. **Induced Maps** Given a continuous map $f : X \to Y$, we define $f^* : C_i(Y;R) \to C_i(X;R)$ as the dual map to f_*. As above, this means:

$$\langle f^*\alpha, c\rangle = \langle \alpha, f_*c\rangle$$

   In particular, if the chain c is a single map $\phi \in \mathrm{Map}(\Delta^i, X)$, then:

$$\langle f^*\alpha, \phi\rangle = \langle \alpha, f \circ \phi\rangle$$

Exercise 4.1. Verify Axiom 2.2.3 (Dimension) from Definition 2.2. That is, show that $H_{i,\mathrm{sing}}(P;R) = 0$ if $i > 0$ and $H_{0,\mathrm{sing}}(P;R) = R$ for a 1 point space P.

Exercise 4.2. Verify Axiom 2.2.4 (Additivity) from Definition 2.2.

4.2 Cellular Homology

Definition 4.3. Let X be a Hausdorff topological space. A **cell map** in X of dimension k is a map $\phi : \bar{D}^k \to X$ from the closed ball to X which is a homeomorphism on the interior of the ball. A **closed cell** is an image of the boundary ∂D^k for some cell map. An **open cell** is the image of the interior D^k for some cell map. □

Definition 4.4. A **cellular decomposition** of a compact Hausdorff space X is a partition of X into finitely many open cells with cell maps $\phi^k_\alpha : \bar{D}^k \to X$ (where k denotes the dimension and α is simply some index) that satisfies the following 2 properties. Let $c^k_\alpha := \phi^k_\alpha(D^k), \bar{c}^k_\alpha := \phi^k_\alpha(\bar{D}^k)$ and $\partial c^k_\alpha := \phi^k_\alpha(\partial D^k)$. Also let $X^k = \cup_{\alpha, j \le k} c^k_\alpha$. The 2 properties are:

1. The boundary ∂c^k_α of each cell c^k_α is contained within X^{k-1}.

2. A subset K of X is closed if and only if $K \cap \bar{c}^k_\alpha$ is closed for all cells $\bar{c}^k_\alpha$. □

The idea here is that a cellular decomposition of X is a set of instructions on how to build X out of standard pieces, namely balls D^k in any dimension, by gluing these pieces together. Property 4.2.1 dictates that the boundary of a piece of dimension k must be glued along the spaces of dimension $k-1$ or lower. Property 4.2.2 dictates that the topology of X is induced as the quotient topology of the cells glued along these maps.

The cellular homology groups $H_i(X;R) = H_{i,\mathrm{CW}}(X;R) := H_i(C^*(X;\mathbb{R}), \partial)$ are defined by:

1. **Chain Groups** The chain group $C_i(X;R)$ is the free R-module generated by the open cells c^i_α of dimension i in any cellular decomposition of X. That is, the chain group is the set of all formal sums:

$$c^i = \sum_\alpha r^i_\alpha c^i_\alpha \quad r^i_\alpha \in R$$

We find it helpful to imagine these chains as collections of k-dimensional sub-manifolds c^i_α sitting inside of X, with little labels r^i_α by elements of R indicating the multiplicity of each of the pieces. Note that we must **choose** an arbitrary cell decomposition: it is a non-trivial result that the homology will not depend on our choice.

2. **Chain Differential** The differential $\partial_i : C_i(X;R) \to C_{i-1}(X;R)$ is defined as so.

First, note that for any c^i_α, the quotient $X^i/(X^i - c^i_\alpha)$ is homeomorphic to $D^i/\partial D^i \simeq S^i$ via the map induced by ϕ^i_α (call this induced map $[\phi^i_\alpha]$). Thus for each c^{i+1}_α and c^i_β we have a map $\chi^\beta_\alpha : S^i \to S^i$ given as the composition of the string of maps:

$$S^i \simeq \partial D^{i+1} \xrightarrow{\phi^{i+1}_\alpha} X_i \xrightarrow{q} X_i/(X_i - c^i_\beta) \simeq c^i_\beta/\partial c^i_\beta \xrightarrow{[\phi^i_\beta]^{-1}} D^i/\partial D^i \simeq S^i$$

Recall that the degree $\deg(\phi)$ of a map $\phi : S^i \to S^i$ is the image of the homotopy class $[\phi] \in \pi_i(S^i)$ under the isomorphism $\pi_i(S^i) \simeq \mathbb{Z}$ where Id is send to 1. Intuitively, the degree $\deg(\chi^\beta_\alpha)$ counts the number of layers of the sphere ∂D^i that the map ϕ^i_α puts onto cell c^{i-1}_β.

The chain differential can now be defined by the following formula. Since $C_{i+1}(X,R)$ is generated by the cells c^{i+1}_α, it suffices to specify $\partial_{i+1}(c^{i+1}_\alpha)$.

$$\partial_{i+1}(c^{i+1}_\alpha) = \sum_\beta \deg(\chi^\beta_\alpha) c^i_\beta$$

As with singular homology, this construction can be dualized to formulate a cellular cohomology $H^i_{CW}(X;R) := H^i(C^*(X;R),\delta)$.

4.3 De Rham Cohomology

Let X be a compact smooth manifold. We can consider the bundles $\Lambda^k X = \Lambda^k(T^*X)$, whose fiber $\Lambda^k X_p$ at each point $p \in X$ is the space of anti-symmetric k-linear maps $T_pX \times \cdots \times T_pX \to \mathbb{R}$ from the tangent space T_pX at p to $\mathbb{R}$, and consider the space of k-forms $\Omega^k(X)$, i.e the space $\Omega^k(X) = \Gamma(\Lambda^k(X))$ of smooth sections of $\Lambda^k(X)$. Recall that there is a natural map $d : \Omega^k(X) \to \Omega^{k+1}(X)$ called the exterior derivative given in coordinates by:

$$d\omega = \sum_I df \wedge dx_I \quad \text{if } \omega = \sum_I f dx_I$$

Here $x_1, \ldots, x_n$ are coordinates in a patch, dx_i are the 1-forms given by the differential of the coordinate functions, and $df = \sum_i \partial_i f dx_i$ is the differential of f. Also:

$$dx_I = dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_k} \text{ with } 1 \leq i_1 \leq \cdots \leq i_k \leq n$$

The De Rham cohomology groups $H^i_{dR}(X) := H^i(C^*(X;\mathbb{R}), \delta)$ are defined by:

1. **Chain Groups** The co-chain group $C^i(X;\mathbb{R})$ is the space of i-forms $\Omega^i(X)$.
2. **Chain Differential** The co-chain differential is $\delta_i : C^i(X;\mathbb{R}) \to C^{i+1}(X;\mathbb{R})$ is given by the exterior derivative $d : \Omega^i(X) \to \Omega^{i+1}(X)$.
3. **Induced Maps** Given a *smooth map* $f : X \to Y$, the induced co-chain map $f^* : \Omega^i(Y) = C^i(Y;\mathbb{R}) \to C^i(Y;\mathbb{R}) = \Omega^i(X)$ is given by pullback.

Exercise 4.3. Show that the wedge product descends to a product on cohomology. That is, show that if $[\alpha] \in H^i_{dR}(X)$ and $[\beta] \in H^j_{dR}(X)$, and α, β are their representative i and j forms, then $[\alpha] \wedge [\beta] := [\alpha \wedge \beta]$ is a well-defined cohomology class, not depending on our choice of α and β representing $[\alpha]$ and $[\beta]$.

5 Additional Properties

In this section we will discuss various important additional properties of homology theories that are not evident from the axioms.

5.1 Important Identities

Theorem 5.1. Universal Coefficients Let X be a topological space and R be any ring. Then there is a short exact sequence:

$$0 \to H_i(X;\mathbb{Z}) \otimes R \to H_i(X;R) \to \mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), R) \to 0$$

In particular, $\mathrm{Tor}(H_{i-1}(X;\mathbb{Z}), \mathbb{R}) = 0$ so that:

$$H_i(X;\mathbb{R}) \simeq H_i(X;\mathbb{Z}) \otimes \mathbb{R}$$

□

Theorem 5.2. Künneth Formula Let X, Y be two topological spaces and F a field. Then:

$$H_k(X \times Y; F) \simeq \bigoplus_{i+j=k} H_i(X;F) \otimes H_j(Y;F)$$

□

5.2 Fundamental Class, Intersection Pairing & Poincare Duality

The intersection product is an extremely important structure on homology, since it transforms some questions in the intersection theory of manifolds into algebraic problems in homology.

Proposition 5.1. *Let Y be a closed (compact, boundary-less) oriented manifold of dimension n. Then $H_n(Y;\mathbb{Z}) \simeq \mathbb{Z}$ and their is a canonical generator $[Y]$ determined by the orientation.*

Definition 5.1. The generator $[Y] \in H_n(Y;\mathbb{Z})$ of Y above is called the **fundamental class** of Y. If $i : Y \to X$ is an embedding of Y into X, then the push-forward $i_*[Y] \in H_n(X;\mathbb{Z})$ is also often referred to as **the fundamental class of** Y.

Proposition 5.2. *Every homology class* $a \in H_i(X;\mathbb{Z})$ *has* $a = i_*[Y]$ *for some oriented* Y *and some embedding* $i : Y \to X$.

Definition 5.2. We define the **intersection pairing** $H_*(X;\mathbb{Z}) \otimes H_*(X;\mathbb{Z}) \to H_*(X;\mathbb{Z})$, usually written as $a \otimes b \mapsto a \cdot b$ or $a \otimes b \mapsto a \cap b$, as so. For any $a \in H_i(X;\mathbb{Z})$ and $b \in H_j(X;\mathbb{Z})$, let Y and Z be closed oriented manifolds of dimension i and j respectively with

Theorem 5.3. Poincare Duality Let X be a closed oriented manifold with $n = \dim(X)$. Then exists an isomorphism:

$$\mathrm{PD} : H_i(X;\mathbb{Z}) \simeq H^{n-i}(X;\mathbb{Z})$$

and its inverse (which we will also call PD) satisfying the following property:

$$\mathrm{PD}(\alpha) \cdot b = \langle \alpha, b \rangle \text{ for all } \alpha \in H^i(X;\mathbb{Z}), \beta \in H_i(X;\mathbb{Z})$$

Remember that $\langle \alpha, b \rangle$ denotes the dual pairing of α and b using the map ϕ discussed in Exercise 3.3. □