Math 55 Section 101 Quiz 2 Solutions

Problem 1 True or False? If true, justify/prove your answer. If false, give a counter-example. All functions are between real numbers, $\mathbb{R} \to \mathbb{R}$.

1.A (2 pts) $f(x) = x^2 + 1$ is injective.
Answer: False. $f(-1) = f(1) = 2$ so the function is many to one.

1.B (2 pts) $g(x) = x^4 - 100$ is surjective.
Answer: False. $-101 \neq g(x)$ for any x because $g(x) = x^4 - 100 \geq -100$ for any x.

1.C (2 pts) $h(x) = -2x + 5$ is bijective.
Answer: True! This function has an inverse given by $h^{-1}(x) = \frac{5-x}{2}$.

1.D (2 pts) A polynomial function $p(x)$ is called *nth order* if the highest power of x that it contains is x^n. For example, $2x^2 + 5$ is 2nd order and $10x^7 + 2x^2 + 1$ is 7th order. True or False: If a polynomial function is $\geq$ 2nd order (that is, 2nd order or higher) then it is *not* injective?
Answer: False. x^3 is a counter-example because it is injective and degree 3.

Problem 2 (1 pt) (1.8 Q 29) Prove that there is no integer n such that $n^3 + n^2 = 100$.
Answer: Proof by contradiction. Suppose for the sake of contradiction that there did exist such an n. Then $n^2(n+1) = 100 = 5^2 \cdot 2^2$. Then either $n^2 = 2^2, n^2 = 5^2$ or $n^2 = 100$, since n^2 must be a square factor of 100. In these cases $n = 2, 5$ or 10. But if we check cases, we see that if $n = 2$ then $n^2(n+1) = 4 \cdot 3 = 12 \neq 100$, if $n = 5$ then $n^2(n+1) = 25 \cdot 6 \neq 100$ and if $n = 10$ then $n^2(n+1) = 100 \cdot 11 \neq 100$. So none of these cases are possible, meaning that no such n could have existed.

Problem 3 (1 pt) Describe a bijection between $S := \{x \in \mathbb{Z} | x > 0\}$ (non-negative integers) and $\mathbb{Z}$.
Answer: Define $f : S \to \mathbb{Z}$ as so. If $m = 2k \in S$ is even or zero, define $f(m) = m/2 = k$. If $m = 2k+1$ is odd, then define $f(m) = -\frac{m+1}{2} = -k$. The inverse is given by the function $g : \mathbb{Z} \to S$ defined as $g(k) = 2k$ for $k \geq 0$ and $g(k) = -2k - 1$ for $k < 0$. Since f has an inverse (with $f \circ g : \mathbb{Z} \to \mathbb{Z}$ and $g \circ f : S \to S$ both the identity) f is a bijection.