Math 535A: Take Home Midterm

Instructions. Please write solutions to 12 of the questions below. You must satisfy the following constraints.

- You may not do more that **four** questions from any one section.
- You must complete at least **five** questions marked as challenge problems.

Please only use facts and methods discussed in class or covered in the book readings (in particular, no materials from Math 540). This assignment is due **in class on 2/28.**

Section 1: Basics

Problem 1. Give an example of a Hausdorff, compact topological space that is not a manifold and prove that it is not a manifold.

Problem 2. Describe an atlas $\mathcal{A}$ on the manifold

$$Y = \{(x, y, z) \ : \ x^4 + y^2 + z^2 = 1\}$$

explicitly as a collection of charts, and verify that the transition maps are smooth.

Problem 3 (Challenge). Consider the following pair of smooth embeddings $f : S^1 \to \mathbb{R}^2$ and $g : S^1 \to \mathbb{R}^2$.

$$f(\theta) = \begin{bmatrix} \cos(2\pi\theta) \\ \sin(2\pi\theta) \end{bmatrix} \qquad \text{and} \qquad g(\theta) = \begin{bmatrix} \cos(-2\pi\theta) \\ \sin(-2\pi\theta) \end{bmatrix}$$

Prove or disprove: there exists a map

$$F : [0, 1] \times S^1 \to \mathbb{R}^2$$

such that $F_0 = f, F_1 = g$ and F_t is an embedding for each $t \in [0, 1]$.

Section 2: Vector Bundles

Problem 4. Give an example of a sub-manifold $Y \subset X$ of a manifold X such that the normal bundle νY is not isomorphic to the trivial bundle, and prove the non-triviality.

Problem 5 (Challenge). Let X be a topological space and let $E \to X$ be a real vector-bundle of rank k. Suppose that E splits as

$$E = L \oplus L \oplus \cdots \oplus L$$

where L is a rank 1 vector bundle over X. Show that $E \otimes E^*$ is isomorphic to the trivial bundle.

Problem 6. Show that every exact sequence of vector-bundles splits. That is, show that given a sequence of vector-bundles

$$0 \to D \xrightarrow{\iota} E \xrightarrow{q} F \to 0$$

where ι is injective, q is surjective and $\ker(q) = \mathrm{im}(q)$, there is an isomorphism

$$E \simeq D \oplus F$$

such that q is identified with projection to F and ι is identified with inclusion of $D \oplus 0$.

Problem 7 (Challenge). This problem is about the homotopy invariance of pullback.

Definition 1. A map $\pi : E \to B$ of topological spaces satisfies the *homotopy lifting property* with respect to a space X if for every pair of continuous maps

$$f : [0,1] \times X \to B \qquad \text{and} \qquad F : X \to E$$

such that $\pi \circ F = f_0$, there is a continuous map

$$H : [0,1] \times X \to E \qquad \text{with} \qquad \pi \circ H = f$$

Definition 2. A *fiber bundle* $\pi : E \to B$ with fiber F is a space E and a continuous map $\pi : E \to B$ such that every point p has a neighborhood U and a diffeomorphism

$$\pi^{-1}(U) \simeq U \times F \qquad \text{sending } \pi^{-1}(x) \text{ to } x \times F$$

Theorem 1. *Every fiber bundle $\pi : E \to B$ satisfies the homotopy lifting property with respect to any manifold X.*

Use the homotopy lifting property to prove that, if $\pi : E \to B$ is a vector-bundle and $f : [0,1] \times X \to B$ is a continuous map, then

$$f_0^* E \simeq f_1^* E$$

Section 3: Special Maps

Problem 8. Let $F : \mathbb{R}^4 \to \mathbb{R}^2$ be the map given by

$$F(w, x, y, z) = (3xy + yz, wx + 5)$$

Show that $F^{-1}(1, 0)$ is an embedded sub-manifold of $\mathbb{R}^4$ and compute its dimension.

Problem 9. Prove or disprove: the image of a map $f : M \to N$ from a closed manifold M to a closed manifold N is an embedded sub-manifold if and only if f is injective and has injective differential $Tf_p : T_pM \to T_pN$ for every $p \in M$.

Problem 10. Let $f : M \to N$ be a smooth map and let $x \in N$ be a regular value. Show that $\Sigma = f^{-1}(x)$ has trivial normal bundle.

Problem 11 (Challenge). Show that a smooth map $f : M \to \mathbb{R}^m$ from a closed n-manifold to $\mathbb{R}^m$ for $n \geq m$ must have a critical point (i.e. a point with differential of rank less than m).

Problem 12. Prove or disprove: the intersection of two embedded sub-manifolds is always an embedded sub-manifold.

Problem 13 (Challenge). Prove that there is no surjective submersion $f : \mathbb{R}P^3 \to T^3$.

Section 4: Lie Groups And Lie Algebras

Problem 14. Prove that the tangent bundle TG of any Lie group G is isomorphic to the trivial bundle.

Problem 15. Let U and V be the following vector-fields on $\mathbb{R}^2$.

$$U = \frac{\partial}{\partial x} + x\frac{\partial}{\partial y} \qquad \text{and} \qquad 2y\frac{\partial}{\partial x} + \frac{\partial}{\partial y}$$

Determine if there are coordinates (s,t) definined in a neighborhood of the origin such that

$$\frac{\partial}{\partial s} = U \qquad \text{and} \qquad \frac{\partial}{\partial t} = V$$

Problem 16. Show that the unitary group $U(n)$ of complex $n \times n$ matrices A satifying $AA^\dagger = I$ is a Lie group and compute its dimension.

Problem 17. Let $S \subset \mathrm{GL}_{2n}(\mathbb{R})$ be the set of invertible $2n \times 2n$ matrices that commute with the matrix

$$J = \begin{bmatrix} 0_n & -I_n \\ I_n & 0_n \end{bmatrix}$$

Show that S is a Lie subgroup of $\mathrm{GL}_{2n}(\mathbb{R})$ and compute its dimension.

Problem 18 (Challenge). Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. Assume that the Lie bracket satisfies

$$[u,v] = 0 \qquad \text{for every } u, v \in \mathfrak{g}$$

Show that G is abelian.

Problem 19 (Challenge). Construct a Lie group action $\rho : G \times T^3 \to T^3$ of some Lie group G on the three torus $T^3 = (\mathbb{R}/\mathbb{Z})^3$ such that

- ρ is transitive on some non-empty open set U, i.e. for any pair $u, v \in U$, we have

$$\rho_g(u) = v \qquad \text{for some } g \in G$$

- ρ is not transitive on T^3, i.e. there are points u and v such that

$$\rho_g(u) \neq v \qquad \text{for any } g \in G$$

Problem 20 (Challenge). Prove or disprove: The centralizer $C_G(S) \subset G$ of a subset $S \subset G$ of a Lie group is a Lie sub-group, for any subset S.

Section 5: Tensors And Flows

Problem 21. Compute the Lie derivative of the following $(1,1)$-tensor T on $\mathbb{R}^2$ with respect to the following vector-field V.

$$T = e^{x+y} \cdot \frac{\partial}{\partial x} \otimes dy \qquad \text{and} \qquad V = x^2\frac{\partial}{\partial y} - e^y\frac{\partial}{\partial x}$$

Problem 22. Let $A : TX \to TX$ and $B : TX \to TX$ be bundle maps, which may be regarded as $(1,1)$-tensors. Let $\Phi : \mathbb{R} \times X \to X$ be a flow generated by a vector-field V.

$$\Phi_t^* A = A \quad \text{for all } t \qquad \text{and} \qquad \mathcal{L}_V B = 2B$$

Show that $\Phi_t^*(AB) = e^{2t} \cdot AB$ where $AB : TX \to TX$ is the composition as bundle maps.

Problem 23 (Challenge). Let X be a smooth manifold and let $\Phi : \mathbb{R} \times X \to X$ be a flow generated by a vector-field V. Suppose that there exists a smooth function

$$f : X \to \mathbb{R} \qquad \text{such that} \qquad f(\Phi_t(x)) > f(x) \text{ for every } t > 0$$

Show that X cannot be a closed manifold.

Problem 24. Show that the diffeomorphism group $\mathrm{Diff}(X)$ acts transitively on any connected manifold X.

Definition 3 (Isotopy of Diffeos)**.** An *isotopy* of diffeomorphisms $f : [0,1] \times X \to X$ is a smooth map such that $f_t : X \to X$ is a diffeomorphism for each $t \in [0,1]$.

Two diffeomorphisms g and h are *isotopic* if there is an isotopy such that $g = f_0$ and $h = f_1$.

Problem 25. Give an example of a closed manifold X and a diffeomorphism $\phi : X \to X$ that is not isotopic to the identity.

Problem 26 (Challenge). Give an example of a closed manifold X and a diffeomorphism $\phi : X \to X$ such that

- ϕ is isotopic to the identity.
- There does not exist a vector-field V whose flow $\Phi : \mathbb{R} \times X \to X$ satisfies $\Phi_1 = \phi$.