Characteristic Classes
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Problem 4-A Show that the Stiefel-Whitney classes of a Cartesian product are given by $w_k(\xi \times \eta) = \sum_{i=0}^{k} w_i(\xi) \times w_{k-i}(\eta)$.

Solution 4-A We note that $\xi \times \eta \simeq p_X^* \xi \oplus p_Y^* \eta$ with $p_X : X \times Y \to X$ and $p_Y : X \times Y \to Y$ being the projection maps, and $\oplus$ the Whitney sum. Now we just note that $w_i(\xi \times \eta) = w_i(p_X^* \xi \oplus p_Y^* \eta) = \sum_{k=0}^{i} w_k(p^* \xi) \cup w_{i-k}(p^* \eta) = \sum_{k=0}^{i} w_k(\xi) \times w_{i-k}(\eta)$. The last identity uses the definition of the product $\times$.

Problem 4-B Prove the following theorem of Stiefel. If $n + 1 = 2^r m$ with m odd, then there do not exist 2^r vector fields on the projective space P^n which are everywhere linearly independent.

Solution 4-B By Theorem 4.5 we know that $w(P^n) = (1 + a)^{n+1}$ for a non-zero section a of $H^1(P^n)$. If $n = 2^r m - 1$ then $w(P^n) = (1 + a)^{2^r m} = (1 + a^{2^r})^m$. The later polynomial has coefficient $m = 1 \mod 2$ in for the a^{2^r} factor, thus $w_{n-2^r+1}(P^n) = w_{2^r(m-1)}(P^n) = w^{2^r}(P^n) \neq 0$. Thus by Proposition 4.4 we know that 2^r global sections cannot exist.

Problem 4-C A manifold M is said to admit a field of tangent k-planes if its tangent bundle admits a sub-bundle of dimension k. Show that P^n admits a field of tangent 1-planes if and only if n is odd. Show that P^4 and P^g6 do not admit fields of tangent 2-planes.

Solution 4-C Again, $w(P^n) = (1 + a)^{n+1}$. If $TP^n = \xi \oplus \eta$ for a line-bundle ξ, then we must have $w(\xi)w(\eta) = w(P^n) = (1 + a)^{n+1}$. $w(\xi)$ and $w(\eta)$ must be order 1 and $n - 1$ respectively (by the axioms of Stiefel-Whitney classes) so it must be the case that $w(\xi) = 1 + a$ or 1.

If n is even, then in the first case, $w(\eta) = (1 + a)^n$ has $w_n(\xi) = \binom{n}{n} a^n = a^n$, and in the latter case $w_n(\xi) = \binom{n+1}{n} a^n = a^n$. This contradicts the fact that for an $n - 1$-bundle such as η, $w_n(\eta) = 0$. Thus such a splitting cannot happen.

If n is odd, then $w_n(P^n) = 0$. This implies that TP^n admits one global section by , whose span defines a 1-plane field.

Likewise, take P^4. Then $w(P^4) = (1+a)^5$ and if $TP^4 = \xi \oplus \eta$ with ξ 2-dimensional then $w(\xi) = 1, 1+a$ or $(1+a)^2$ and $w(\eta) = (1+a)^5, (1+a)^4$ or $(1+a)^3$. In the first two cases, there is a non-zero a^4 coefficient and in the last case there is a non-zero a^3 coefficient, even though η must be a 2-bundle. So this isn't possible. Finally, given P^6 we see that using a similar splitting $TP^6 = \xi \oplus \eta$ we know that $w(\eta) = (1 + a)^7, (1 + a)^6$ or $(1 + a)^5$ and η must be dimension 4. The same argument then shows that in the first two cases there is a non-zero a^6 coefficient and in the last a non-zero a^5 coefficient, which contradicts the fact that η is a 4-bundle.

Problem 4-D If the n-dimensional manifold M can be immersed in $\mathbb{R}^{n+1}$ show that each $w_i(M)$ is equal to the i-fold cup product $w_1(M)^i$. If P^n can be immersed in $\mathbb{R}^{n+1}$ show that n must be of the form $2^r - 1$ or $2^r - 2$.

Solution 4-D In the above case, we see that $w(TM) = \bar{w}(NM)$. Furthermore, $w_1(TM) = \bar{w}_1(NM)$ and since NM is 1-dimensional, $w_i(NM) = 0$ for $i \geq 2$. Then Taylor expansing $w(TM) = (1 + w(NM))^{-1} = (1 + w_1(NM) + \dots)^{-1}$ shows us that $w_i(TM) = w_1(NM)^i + \dots = w_1(TM)^i$ where $\dots$ are terms $w_i(NM)$ for $i \geq 2$.

Applying this, we see that if P^n can be embedded in this way then $w(TP^n) = (1 + a)^{n+1} = 1 + \sum_{i=1}^{n} w_1(TM)^i$. And we see that when $n = 2^r m - 1$ for some odd m, then $(1 + a)^{n+1} = (1 + a^{2^r})^m$. If m is 1, then $w_1(TP^n) = 0$ makes $w(TP^n)$ satisfy the necessary condition set out by the above statement. Likewise, if $m \neq 1$ but $r = 0$ (thus $n = m - 1 = 2^r - 2$ for some r) then $(1 + a)^m = 1 + \sum a$ and $w_1(TP^n) = a$ makes $w(TP^n)$ satisfy the necessary condition set out by the above statement. Otherwise, $(1 + a)^{n+1}$ has a 0 a coefficient and a non-zero m coefficient, so it cannot satisfy the above statement.

Problem 4-E Show that the set C_n of all unoriented cobordism classes of smooth closed n-manifolds can be made into an abelian group. This *cobordism group* G_n is finite by Proposition 4.11, and is also a module over $\mathbb{Z}/2\mathbb{Z}$. Using the manifolds $P^2 \times P^2$ and P^4, show that G_4 contains at least four distinct elements.

Solution 4-E The addition operation on C_n is disjoint union $\sqcup$. We see that if $[M_1] = [N_1]$ and $[M_2] = [M_2]$ (where M_i and N_i are closed n-manifolds and $[\cdot]$ is the cobordism class) then $[M_1 \sqcup M_2] = [N_1 \sqcup N_2]$ since the disjoint unions are the boundary of the disjoint union of the manifold with boundary $M_1 \sqcup N_1$ and the manifold with boundary $M_2 \sqcup N_2$. Commutativity and associativity follow from the fact that $\sqcup$ has these properties on manifolds. The identity is given by the class of manifolds $[M]$ which are already the boundary of some manifold with boundary. Last, every element is its own inverse, since $M \sqcup M$ is the boundary of $M \times [0, 1]$ and so is in the identity class.

Now observe that $w(P^4) = (1 + a)^5$ and $w(P^2 \times P^2) = (1 + a)^6$. They are non-cobordant since their Stiefel-Whitney classes are different and they are non-zero since their Stiefel-Whitney classes are non-zero. So $[0], [P^4], [P^2 \times P^2]$ and $[P^2 \times P^2] + [P^4]$ are all distinct cobordism classes.

Problem 5-A Show that the Grassmann manifold $G_n(\mathbb{R}^{n+k})$ can be made into a smooth manifold as follows: a function $f : G_n(\mathbb{R}^{n+k}) \to \mathbb{R}$ belongs to the collection F of smooth real valued functions if and only if $f \circ q : V_n(\mathbb{R}^{n+k}) \to \mathbb{R}$ is smooth.

Solution 5-A It suffices to show that the coordinate patches, coordinate functions and associated transition constructed on p. 58-59 are smooth by this definition. Let $X_0 \in G_n(\mathbb{R}^{n+k})$, let $F \in \mathbb{R}^{n(n+k)}$ be a frame of X_0 and define $U(X_0) \subset G_n(\mathbb{R}^{n+k})$ as $U(X_0) := \{Y \subset G_n(\mathbb{R}^{n+k}) | Y \cap X_0^\perp = \emptyset\}$. Note that $q^{-1}U(X_0) = \{G \in V_n(\mathbb{R}^{n+k} | \Pi G \neq 0\}$ where G is seen as $n \times (n + k)$ matrix and Π is the projection to X_0, so $U(X_0)$ is open in the quotient topology.

Define the coordinate function $\phi_F : U(X_0) \to \mathrm{Hom}(X_0, X_0^\perp) \simeq \mathbb{R}^{n(n+k)}$ as the map sending Y to the map $T(Y) : X_0 \to X_0^\perp$ with graph Y. If $G \in q^{-1}U(X_0)$ and we again interpret G as a matrix (using the basis F for X_0 and a basis $F^\perp$ of $X_0^\perp$ as the basis for $\mathbb{R}^{n+k}$) then the matrix can be written as:

$$M_G = \Pi^\perp G(\Pi G)^{-1} 1_{X_0} =: T(Y)$$

Note that $T(Y)$ is invariant under the right action by $GL_n(\mathbb{R})$, so it only depends on our choice of Y. Here Π and $\Pi^\perp$ are the projection matrices to X_0 and $X_0^\perp$ respectively. $T(Y)$ clearly satisfies $\Pi(1_{X_0}+M_G)1_{X_0} = 1_{X_0}$ and $\mathrm{Graph}(T(Y)) = \mathrm{span}(x + T(Y)x) = \mathrm{col}(G) = Y$ where col is the column space. The map $\phi_F \circ q$ is precisely the map $G \mapsto T(Y)$, and by this expression it is smooth on the chosen open set, since ΠG is invertible on $U(X_0)$. The transition functions $\phi_H \phi_F^{-1} : \mathrm{Hom}(X_0, X_0^\perp) \to \mathrm{Hom}(Y_0, Y_0^\perp)$ is given by $M_F \mapsto \Pi_Y^\perp C_X^Y(1_X + M_F)$. Here C_X^Y is the change of basis from $F, F^\perp$ to $G, G^\perp$. This is also evidently smooth in the matrix M_F, so the transition functions are smooth.

Problem 5-B Show that the tangent bundle τ of $G_n(\mathbb{R}^{n+k})$ is isomorphic to $\mathrm{Hom}(\gamma^n(\mathbb{R}^{n+k}), \gamma^\perp)$ where $\gamma^\perp$ is the normal bundle to $\gamma^n(\mathbb{R}^{n+k})$ in ϵ^{n+k}. Now consider a smooth manifold $M \subset \mathbb{R}^{n+k}$. If $\bar{g} : M \to G_n(\mathbb{R}^{n+k})$ denotes the generalized Gauss map show that $D\bar{g} : DM \to DG_n(\mathbb{R}^{n+k})$ gives rise to a cross-section of $\mathrm{Hom}(\tau M, \mathrm{Hom}(\tau M, \nu)) \simeq \mathrm{Hom}(\tau M \otimes \tau M, \nu)$, the second fundamental form of M.

Solution 5-B We have a naturally defined map $\mathrm{Hom}(\gamma^n, \gamma^\perp) \to \tau$ given fiber-wise at $X \in G_n(\mathbb{R}^{n+k})$ by $T_X \in \mathrm{Hom}(\gamma^n, \gamma^\perp) \mapsto \frac{d}{ds}((1 + sT_X)X)_{s=0} \in \tau_X$. The image of a smooth section under this map clearly varies smoothly with X and the fact that it's fiber-wise bijective follows from an examination in the chart $U(X)$. So this is a bundle isomorphism.

Now examining the Gauss map, we see that $\bar{g}$ is covered by the bundle map $g \oplus g^\perp : \tau \oplus \nu \to \gamma_n \oplus \gamma^\perp$ and the differential $D\bar{g} : \tau M \to \tau G_n(\mathbb{R}^{n+k}) \simeq \mathrm{Hom}(\gamma^n, \gamma^\perp)$. This thus yields a fiber-linear morphism $\sigma : \tau M \to \mathrm{Hom}(\tau M, \nu)$ via $(p, v) \mapsto (g^\perp)^{-1}_{\bar{g}(p)} \circ D\bar{g}_p(v) \circ g_p$. This is precisely a smooth section of the desired bundle.

Problem 5-C Show that $G_n(\mathbb{R}^m)$ is diffeomorphic to the smooth manifold consisting of all $m \times m$ symmetric idempotent matrices of trace n. Alternatively show that the map $(x_1, \ldots, x_n) \mapsto x_1 \wedge \cdots \wedge x_n$ from $V_n(\mathbb{R}^m)$ to the exterior power $\wedge^n(\mathbb{R}^m)$ gives rise to a smooth embedding of $G_n(\mathbb{R}^m)$ in $G_1(\mathbb{R}^{\binom{m}{n}})$.

Solution 5-C For the first part, the diffeomorphism clearly must be defined as $M \mapsto \mathrm{col}(M)$, the column space of M (equivalently, the row-space). The inverse sends an n-plane X to the unique symmetric M_X with 1-eigenspace K and 0 eigenspace $K^\perp$. This matrix is unique because (viewing the matrices as bilinear forms) under any orthogonal change of basis splitting $\mathbb{R}^m$ into $K \oplus K^\perp$ the matrices must be exactly the matrix with n 1's on the K diagonal and $m - n$ 0's on the $K^\perp$ diagonal.

The map is smooth and smoothly invertible in coordinate charts $U(X)$. Indeed, in the basis where M_X is the $1, 0$ matrix described before (let's call it I_n for now), the map can just be written as $T(Y) \mapsto S^T I_n S$ where $S = gs([1_n + T(Y)|B(X^\perp)]$ where $1_n + T(Y)$ is as in Problem 5-A, $B(X^\perp)$ is a matrix populated by an arbitrary basis of $X^\perp$ and $gs(\cdot)$ applies the Graham-Schmidt process to the matrix column-by-column.

These are all smoothly dependent on $T(Y)$ so the expression given is also smooth. The inverse is likewise given by $M \mapsto M_{O(M)}$ where $O(M)$ is the unique $n \times m$ matrix satisfying $O(M)^T M O(M) = 1_n$ and $\Pi_X O(M) = 1_n$. $M_{O(M)}$ corresponds to M_G as written in M_G[1] Everything here again smoothly depends on the idempotent symmetric M, so this establishes that the map $M \mapsto \mathrm{col}(M)$ is indeed a diffeomorphism.

For the second part, observe that $\phi : V_n(\mathbb{R}^m) \to V_1(\wedge^n(\mathbb{R}^m)) = \wedge^n(\mathbb{R}^m)$ given by $(x_1, \dots, x_n) \mapsto x_1 \wedge \cdots \wedge x_n$ is smooth and factors through both quotients, since applying any $T \in GL_n$ to $(x_1, \dots, x_n)$ via right-multiplication (i.e map (x_i) to $M(x_i) = (\sum_i m_{ij} x_j)$ changes $x_1 \wedge \cdots \wedge x_n$ by a multiplicative factor $\det M$). So the map must be smooth on the quotients.

Problem 5-D Show that $G_n(\mathbb{R}^{n+k})$ has the following symmetry property. Given any two n-planes $X, Y \subset \mathbb{R}^{n+k}$ there exists an orthogonal automorphism of $\mathbb{R}^{n+k}$ which interchanges X and Y. [Whitehead, 1961] defines the angle $\alpha(X, Y)$ between n-planes as the maximum over all unit vectors $x \in X$ of the angle between x and Y. Show that α is a matrix for the topological space $G_n(\mathbb{R}^{n+k})$ and show that $\alpha(X, Y) = \alpha(Y^\perp, X^\perp)$.

Solution 5-D Without loss of generality, assume that $X \cap Y = 0$ and that $X, Y \in G_n(\mathbb{R}^{2n})$. We can make these assumptions by fixing the subspace $\mathrm{span}(X \cap Y, (X \cup Y)^\perp)$, so that all of our rotations take place in a subspace $V \subset \mathbb{R}^{n+k}$ where these assumptions hold for $X \cap V$ and $Y \cap V$. In this regime, let $e_1, \dots, e_n$ be an ortho-basis for X and $f_1, \dots, f_n$ be a complimentary basis. Break $\mathbb{R}^{2n}$ into $\oplus_{i=1}^n \mathrm{span}(e_i, f_i)$. Note that each of these subspaces must intersect Y on a one-dimensional subspace precisely (if the intersection is 0 this violates the dimension of Y and if it's two the $X \cap Y$ is non-trivial). Furthermore, the intersection lines collectively span Y. Now on each $\mathrm{span}(e_i, f_i)$ let U_i be the orthogonal transformation exchanging $\mathrm{span}(e_i)$ and $\mathrm{span}(e_i, f_i) \cap Y$ via reflection across the line bisecting the angle between them and at an angle < 180 degrees from $\mathrm{span}(e_i)$. Then the orthogonal transformation $\prod_i U_i$ on $\mathbb{R}^{2n}$ exchanges Y and X by exchanging basis 1-dimensional subspaces.

Now let's look at the metric α. We verify the metric axioms. It is clear that the angle of a vector $x \in X$ with Y is 0 if and only if $x \in Y$, so if $\alpha(X, Y) = 0$ then $X \subset Y$ thus $X = Y$. Furthermore, the angle is manifestly non-negative $\alpha(X, Y) > 0$ if $X \neq Y$. By the above argument, there is an angle preserving linear map exchanging any X and Y, so $\alpha(X, Y) = \alpha(Y, X)$. Last, $\alpha(X, Z)$

Problem 5-E Let ξ be an $\mathbb{R}^n$-bundle over B.

Problem 5-E(1) Show that there exists a vector bundle η over B with $\xi \oplus \eta$ trivial if and only if there exists a bundle map $\xi \to \gamma^n(\mathbb{R}^{n+k})$ for large k. ξ is then called *finite type*.

Solution 5-E(1) If there is such a map $\psi : \xi \to \gamma^n(\mathbb{R}^{n+k})$ then it comes from a map $\phi : \xi \to \mathbb{R}^{n+k}$ which is fiber-wise linear. This map can be interpreted as a bundle map $\xi \to \epsilon^{n+k}$, the trivial $n+k$-bundle over B. We can construct a normal bundle η as the perpendicular sub-bundle to $\psi(\xi) \subset \epsilon^{n+k}$ where ϵ^{n+k} is given a metric by $\mathbb{R}^{n+k}$. Conversely, if there exists a summand η with $\xi \oplus \eta$ trivial then there exists a fiber-wise

[1] Apologies for using M twice here.

linear map $B \times \mathbb{R}^{n+k} \to \mathbb{R}^{n+k}$ which is obvious and we can just get the desired map by composition with the inclusion $\xi \to \xi \oplus \eta$.

Problem 5-E(2) Now assume B is normal. Show that ξ has finite type if and only if B is covered by finitely many open sets $U_1, \ldots, U_r$ with $\xi|U_i$ trivial.

Solution 5-E(2) If there exist such U_i, then we can construct a map to $\gamma^n(\mathbb{R}^{rn})$ using the proof in Lemma 5.3. This requires normality. Conversely, if ξ has finite type then we can pull back trivializations of $\gamma^n(\mathbb{R}^{n+k})$ of which there are finitely many, since $G_n(\mathbb{R}^{n+k})$ is compact.

Problem 5-E(3) If B is paracompact and has finite covering dimension, show (using the argument of 5.9) that every ξ over B has finite type.

Solution 5-E(3) Finite covering dimension implies that the $U(S)$ costructed in the argument of Lemma 5.9 are empty for $|S| > k$ and k sufficiently high, so the construction will only require a finite $\mathbb{R}^N$ to embed into. That is, there will be finitely many sets $U_1, \ldots, U_k$ which cover B on which the bundle is trivial, and the bundle can thus be given a bundle map $\xi \to \mathbb{R}^{nk}$ via $(p, v) \mapsto \oplus_{i=1}^k \lambda_i(p)\pi_i(p, v)$ where π_i is the projection $U_i \times \mathbb{R}^n \to \mathbb{R}^n$.

Problem 5-E(4) Using Stiefel-Whitney classes show that γ^1 over $\mathbb{R}P^\infty$ is not of finite type.

Solution 5-E(4) Any complimentary bundle η would have to have $w(\eta) = \bar{w}(\gamma^1) = \sum_{i=0}^\infty a^i$. But this is impossible for any finite-dimensional bundle since the Stiefel-Whitney class must vanish at cohomology of higher dimension than its fibers.

Problem 6-A Show that a CW-comple is finite if and only if it's underlying space is compact.

Solution 6-A Suppose that X is a finite CW complex. Then there is a continuous quotient map $\cup_\alpha C_\alpha \to X$ given by the gluing process. $\cup_\alpha C_\alpha$ is compact, so its image is as well. Now suppose that X is an infinite CW complex. Then there are two possibilities: X has infinitely many cells in some dimension k or X has one cell of dimension k_i for some sequence $k_i \to \infty$ and each sub-complex X_k of all cells with dimension $\leq k$ is finite.

In the first case, observe that every sub-complex X_k is closed, so we can assume without loss of generality that there are infinitely many cells of largest dimension k. Then the interior of each cell is open and the collection of these interiors forms a countably infinite set of disjoint opens, so X_k is not compact. Thus X cannot be either. Let C_{k_i} be a cell in each dimension k_i and pick points $p_i \in C_i$. The sequence p_i cannot have a limit point: such a limit point, say p, would be in a finite sub-complex X_k and thus would be separated from all but finitely many p_i by the open $X \setminus X_k$, contradicting the fact that it was a limit point. So X is not compact in this case either. This concludes the proof.

Problem 6-B Show that the restriction homomorphism $i^* : H^p(G_n(\mathbb{R}^\infty)) \to H^p(G_n(\mathbb{R}^\infty))$ is an isomorphism for $p < k$. Any coefficient group may be used.

Solution 6-B Via Corollary 6.7 we see that the inclusion $G_n(R^{n+k}) \to G_n(\mathbb{R}^\infty))$ is onto on the $r \le k$ cells. This is because this inclusion takes cells to cells and the number of cells is the number of partitions of r into n integers less than or equal to k (which is always satisfied when $r < k$). Thus there is an isomorphism of chain complexes $C_*(G_n(\mathbb{R}^\infty)) \simeq C_*(G_n(\mathbb{R}^{n+k}), \Lambda)$ if we restrict to the complex for $* \le k$. But this induces an isomorphism of the dual chain complexes as well for these dimensions, thus an isomorphism of homology for $* < k$ (we do not get the kth cohomology necessarily because the k-th cohomology relies on the content of the C^{k+1} chain).

Problem 6-C Show that the correspondence $f : X \to \mathbb{R}^1 \oplus X$ defines an embedding of the Grassmanian manifold $G_n(\mathbb{R}^m)$ into $G_{n+1}(\mathbb{R}^1 \oplus \mathbb{R}^m) = G_{n+1}(\mathbb{R}^{m+1})$ and that f is covered by a bundle map $\epsilon^1 \oplus \gamma^n(\mathbb{R}^m) \to \gamma^{n+1}(\mathbb{R}^{m+1})$. Show that f carries the r-cell of $G_n(\mathbb{R}^m)$ which corresponds to the partition $i_1, \dots, i_s$ of r onto the r-cell of $G_{n+1}(\mathbb{R}^{m+1})$ which corresponds to the same partition $i_1, \dots, i_s$.

Solution 6-C This correspondence clearly defined an injective map $f : G_n(\mathbb{R}^m) \to G_{n+1}(\mathbb{R}^{m+1})$, so we only need to check that it is smooth. But this is clear, since on the coordinate patches $U(X)$ defined in ch. 5 the map takes the form $M \in \mathrm{Hom}(X, X^\perp) \mapsto [M|0] \in \mathrm{Hom}(X \oplus \mathbb{R}, X^\perp)$. To check that it is covered by a bundle map, we recall that $\gamma^n(\mathbb{R}^m)$ was defined as a sub-bundle of ϵ^m. The map f is accompanied by a natural bundle map $\epsilon^m \to \epsilon^{m+1}$ given by the inclusion $\mathbb{R}^m \to \mathbb{R}^{m+1}$. Furthermore, this bundle map maps the sub-bundle $\gamma^n(\mathbb{R}^m) \to \gamma^{n+1}(\mathbb{R}^{m+1})$, due to the construction of these sub-bundles and the definition of f. The orthocompliment of the embedded fiber of γ^n in γ^{n+1} is just the constant $\mathbb{R}$ sub-space in $\epsilon^{m+1} = \epsilon^m \oplus \mathbb{R}$. So γ^{n+1} splits as $\epsilon^1 \oplus \gamma^n$ along the embedding f, and we get a covering bundle map $\epsilon^1 \oplus \gamma^n \to \gamma^{n+1}$ of f.

Finally, to argue the fact about cell's, just observe that an element X of Schubert cell $e(\sigma)$ corresponding to $\sigma = (\sigma_1, \dots, \sigma_n)$ has $\dim(X \cap \mathbb{R}^{\sigma_i}) = \dim(X \cap \mathbb{R}^{\sigma_i - 1}) + 1 = i$ by definition. Then $f(X) = \mathbb{R} \oplus X$ satisfies $\dim(f(X) \cap \mathbb{R}^{\sigma_i+1}) = \dim(X \cap \mathbb{R}^{\sigma_i}) + 1 = i + 1$ and $\dim(f(X) \cap \mathbb{R}^1) = 1$. This implies that it has the Schubert symbol $(1, \sigma_1 + 1, \sigma_2 + 1, \dots)$ and the partition vector $(0, \sigma_1, \sigma_2, \dots, \sigma_n)$ which is the same as before.

Problem 6-D Show that the number of distinct Stiefel-Whitney numbers is equal to $p(n)$ (the number of partitions of n).

Solution 6-D The Stiefel-Whitney numbers correspond bijectively with their indices (r_i), which are defined as non-negative integers such that $\sum_i i r_i = n$ and $1 \le i \le n$. These are in bijection with partitions of n via the correspondence (r_i) maps to the partition of n into r_1 1's, r_2 2's and so on. You can recover (r_i) from a partition by counting the number of instances of i in a partition and setting r_i equal to that.

Problem 6-E Show that the number of r-cells in $G_n(\mathbb{R}^{n+k})$ is equal to the number of r-cells in $G_k(\mathbb{R}^{n+k})$. Or show that they are isomorphic as cell complexes.

Solution 6-E By Corollary 6.7 it's sufficient to show that the number of partitions of r into $\leq n$ integers each $\leq k$ if the same as the number of partitions of r into $\leq k$ integers each $\leq n$. This follows immediately from the fact that each such partition corresponds to an $n \times k$ Young tableaux: a stack of $\leq n$ rows of boxes, with rows of non-increasing in lengths each $\leq k$ and a total of r boxes. Transposing this stack (swapping rows and columns) makes a Young tableaux with n and k switched, and this operation is obviously invertible. This establishes the desired bijection.

The proof of the latter statement is much more involved. We have already shown that the map $\rho : G_n(\mathbb{R}^{n+k}) \to G_k(\mathbb{R}^{n+k})$ which maps an n-plane to the perpendicular k-plane is a diffeomorphism. We will now show that it is also a cell map. Let $\mathbb{R}^0 \subset \mathbb{R}^1 \subset \mathbb{R}^2 \subset \dots$ be the sequence of spaces with which the cell structure on $G_n(\mathbb{R}^{n+k})$ is constructed. On $G_k(\mathbb{R}^{n+k})$, impose the cell-structure using the perpendicular sequence $(\mathbb{R}^{n+k})^\perp \subset (\mathbb{R}^{n+k-1})^\perp \subset \cdots \subset (\mathbb{R}^0)^\perp$. We will show that ρ sends a cell $e(\sigma) \subset G_n(\mathbb{R}^{n+k})$ of dimension $d(\sigma) = r$ corresponding to a sequence $\sigma = (\sigma_i)$ to a dual cell $e(\sigma^*) \subset G_k(\mathbb{R}^{n+k})$ with $d(\sigma^*) = r$ as well. The correspondence $\sigma \to \sigma^*$ will be bijective by construction so this will show that this map carries the cells of $G_n(\mathbb{R}^{n+k})$ diffeomorphically onto those of $G_k(\mathbb{R}^{n+k})$.

To see this cell construction, observe the following. If $\dim(X \cap \mathbb{R}^m) = j$, then $\dim(X^\perp \cap (\mathbb{R}^m)^\perp) = k + j - m$. This implies that $\dim(X \cap \mathbb{R}^m) = \dim(X \cap \mathbb{R}^{m+1})$ if and only if $\dim(X^\perp \cap (\mathbb{R}^m)^\perp) = \dim(X^\perp \cap (\mathbb{R}^{m+1})^\perp) + 1$. Thus, the sequence of spaces $X^\perp \cap (\mathbb{R}^m)^\perp$ decrease in index at every index $m = \sigma_i'$ where σ_i' is the "complimentary sequence" to σ_i, i.e the sequence of length k such that $\{\sigma_i\} \cup \{\tilde{\sigma}_i\} = \{1, \dots, n+k\}$. If we reverse this sequence of spaces so that the $(\mathbb{R}^m)^\perp$ ascend in dimension, then the resulting sequence jumps dimension at $\sigma_i^* = n + k + 1 - \tilde{\sigma}_{k+1-i}$. Note here that we will also use below the sequence $\tilde{\sigma}_i^* = n + k + 1 - \sigma_{n+1-i}$.

So ρ maps $e(\sigma)$ into $e(\sigma^*)$. Note that this process is reversible, i.e taking $(X^\perp)^\perp$ and $(\sigma^*)^*$ yields again σ. Now we just have to argue that $d(\sigma^*) = r$. Once we have done so, the fact that ρ is surjective and invertible will imply that ρ must map $e(\sigma)$ diffeomorphically onto $e(\sigma^*)$. Using all of the previous discussion, we see that:

$$\sum_{i=1}^{n} \sigma_i - i = \sum_{i=1}^{n} \sigma_i - \frac{n(n+1)}{2} = d(\sigma); \sum_{i=1}^{n} \sigma_i = \frac{n(n+1)}{2} + d(\sigma)$$

$$\sum_{i=1}^{k} \sigma_i^* + \sum_{i=1}^{n} \tilde{\sigma}_i^* = \frac{(n+k)(n+k+1)}{2}; \sum_{i=1}^{n} \tilde{\sigma}_i^* = n(n+k+1) - \sum_{i=1}^{n} \sigma_i = n(n+k+1) - \frac{n(n+1)}{2} - d(\sigma)$$

$$d(\sigma^*) = \sum_{i=1}^{k} \sigma_i^* - \frac{k(k+1)}{2} = \frac{(n+k)(n+k+1)}{2} - n(n+k+1) + \frac{n(n+1)}{2} + d(\sigma) - \frac{k(k+1)}{2}$$
$$= \frac{n(n+1)}{2} + nk + \frac{k(k+1)}{2} - n(n+k+1) + \frac{n(n+1)}{2} + d(\sigma) - \frac{k(k+1)}{2} = d(\sigma)$$

Improving this, we can even argue that σ^* is the Schubert symbol corresponding to the conjugate partition of σ. To do this, observe that there $(n+k+1-\sigma_i) - (n+k+1-\sigma_{i+1}) + 1 = (\sigma_{i+1} - (i+1)) - (\sigma_i - i)$

is just one more than the number of blocks by which the ith row and the $i+1$th row of the Schubert cell's Young diagram differ. Thus the sequence of integers greater than $(n+k+1-\sigma_{i+1})$ and less than $(n+k+1-\sigma_i)$ that is in σ^* will correspond to a series of rows in the Young tableaux of σ^* of constant length equal to the number of missing gaps of numbers between 1 and $n+k$ in the sequence $n+k+1-\sigma_{k-i}$. But every But every such gap corresponds to a row to the Young tableaux of σ of larger length than the one below it, thus an increase in the length of the rows of the conjugate. So starting at the beginning of the sequence σ^*, this reasoning implies that the resulting constructed sequence will reproduce the conjugate of σ.

Problem 7-A Identify explicitly the cocycle in $C^r(G_n) \simeq H^r(G_n)$ which corresponds to the Stiefel-Whitney class $w_r(\gamma^n)$.

Solution 7-A First recall that $H^k(G_n(\mathbb{R}^\infty)) \simeq H^k(G_n(\mathbb{R}^{n+k}))$ cell-wise via the inclusion $G_n(\mathbb{R}^{n+k}) \to G_n(\mathbb{R}^\infty)$. So it suffices to compute $w_k(\gamma^n(\mathbb{R}^{n+k}))$. Furthermore, recall that we have a morphism $f : \epsilon^1 \oplus \gamma^n(\mathbb{R}^m) \to \gamma^{n+1}(\mathbb{R}^{m+1})$ which takes a Schubert cell $e(\sigma)$ corresponding to a partition σ to the cell corresponding to the same partition. f also satisfies $f^* w_i(\gamma^{n+1}(\mathbb{R}^{m+1})) = w_i(\epsilon^1 \oplus \gamma^n(\mathbb{R}^m)) = w_i(\gamma^n(\mathbb{R}^m))$ by naturality. By iterative composition we can get morphisms $f_j : \epsilon^j \oplus \gamma^n(\mathbb{R}^m) \to \gamma^{n+j}(\mathbb{R}^{m+j})$ such that $f_j^* w_i(\gamma^{n+j}(\mathbb{R}^{m+j})) = w_i(\gamma^n(\mathbb{R}^m))$ and such that $e(\sigma)$ is send to $e(\sigma)$ if σ is a partition.

Now observe that $f_j^* w_k(\gamma^n(\mathbb{R}^{n+k})) = w_k(\gamma^{n-j}(\mathbb{R}^{n+k-j})) = 0$ if $k > n-j$ by the dimensionality axiom of S-W classes. But the cells in the image of $G_{n-j}(\mathbb{R}^{n+k-j}) \to G_n(\mathbb{R}^{n+k})$ are precisely the cells $e(\sigma)$ with σ a partition of k into $\leq n-j < k$ integers each $\leq k$. This implies that $w_k(\gamma^n(\mathbb{R}^{n+k}))$ is 0 on all k-cells $e(\sigma)$ with partitions σ into fewer than k integers. But there is only one partition of k that doesn't satisfy this property, the cell $e(1,\dots,1)$ for the partition of k into k 1's. So in order to be non-zero, $w_k(\gamma^n(\mathbb{R}^{n+k}))$ must be a non-zero multiple of $e(1,\dots,1)^*$, and there is only one such multiple because we are in $\mathbb{Z}/2$ homology. So $w_k(\gamma^n(\mathbb{R}^\infty)) = e(1,\dots,1)^*$, the dual cell corresponding to the partition of k into k 1's.

Problem 7-B Show that the cohomology algebra $H^*(G_n(\mathbb{R}^{n+k}))$ over $\mathbb{Z}/2$ is generated by the Stiefel-Whitney classes $w_1,\dots,w_n$ of γ^n and the dual classes $\bar{w}_1,\dots,\bar{w}_k$ subject only to the $n+k$ defining relations $w\bar{w} = 1$.

Solution 7-B

Problem 7-C Let ξ^m and η^n be vector bundles over a paracompact base space. Show that the Stiefel-Whitney classes of the tensor product $\xi^m \otimes \eta^m$ can be computed as follows. If the fiber dimensions m and n are both 1, then $w_1(\xi^1 \otimes \eta^1) = w_1(\xi^1) + w_1(\eta^1)$. More generally there is a universal formula of the form:

$$w(\xi^m \otimes \eta^n) = p_{m,n}(w_1(\xi^m),\dots,w_m(\xi^m), w_1(\eta^n),\dots,w_n(\eta^n))$$

Here $p_{m,n}$ is a polynomial in the elementary symmetric polynomials $\sigma_1, \ldots, \sigma_m$ in the variables $t_1, \ldots, t_m$ and $\sigma'_1, \ldots, \sigma'_n$ in the variables $t'_1, \ldots, t'_n$ defined as:

$$p_{m,n}(\sigma_1, \ldots, \sigma_m, \sigma'_1, \ldots, \sigma'_n) = \prod_{i,j=1,1}^{m,n} (1 + t_i + t'_j)$$

Solution 7-C First let ξ^m and η^n be any pair of bundles over B. Consider the bundles $p_m^*\gamma^m, p_n^*\gamma^n$ and $p_m^*\gamma^m \otimes p_n^*\gamma^n$ over $G_m\mathbb{R}^\infty \times G_n\mathbb{R}^\infty$. By universality we have bundle maps $f : \xi^m \to \gamma^m, g : \eta^n \to \gamma^n$, and to get maps (which we will also call f and g) $p_m^*\gamma^m$ and $p_n^*\gamma^n$ we define them by $(b, v) \mapsto (\bar{f}(b) \times \bar{g}(b), f(v))$ and $(b, v) \mapsto (\bar{f}(b) \times \bar{g}(b), g(v)))$. We also get a map $f \otimes g : \xi^m \otimes \eta^n \to p_m^*\gamma^m \otimes p_n^*\gamma^n$ which is $\bar{f} \times \bar{g}$ on the fibers.

Now observe that $H^*(G_m \times G_n, \mathbb{Z}/2) = H^*(G_m, \mathbb{Z}/2) \otimes H^*(G_n, \mathbb{Z}/2)$ by the Kunneth formula. In particular, the Stiefel Whitney classes $w_i(\gamma^m) \times 1$ and $1 \times w_i(\gamma^n)$ of $p_m^*\gamma^m$ and $p_n^*\gamma^n$ generate the cohomology ring of $G_m \times G_n$, so there is a finite polynomial $p_{m,n}$ in $w_i(\gamma^m) \times 1$ and $1 \times w_i(\gamma^n)$ which is equal to $w(p_m^*\gamma^m \otimes p_n^*\gamma^n)$. But then observe that:

$$w(\xi^m \otimes \eta^n) = (f \times g)^* w(p_m^*\gamma^m \otimes p_n^*\gamma^n) = (f \times g)^* p_{m,n}(w_i(\gamma^m) \times 1, 1 \times w_i(\gamma^n))$$

$$= p_{m,n}(f^* w_i(\gamma^m), g^* w_i(\gamma^n)) = p_{m,n}(w_i(\xi^m), w_i(\eta^n))$$

Thus there is such a polynomial relation between the S-W classes of $\xi^m \otimes \eta^n, \xi^m$ and η^n.

Furthermore, observe that this universal polynomial relation must be unique. Otherwise, we would have two polynomials $p_{m,n}$ and $q_{m,n}$ with $p_{m,n}(w_i(\gamma^m), w_i(\gamma^n)) - q_{m,n}(w_i(\gamma^m), w_i(\gamma^n)) = 0$. This would imply a polynomial relation between the $w_i(\gamma^m), w_i(\gamma^n)$, which would contradict the fact that $H^*(G_n, \mathbb{Z}/2) \otimes H^*(G_m, \mathbb{Z}/2)$ is the free $\mathbb{Z}/2$ polynomial ring generated over these variables (since it it the tensor product of the free polynomial rings over the $w_i(\gamma^m)$ and $w_i(\gamma^n)$ respectively). Note that this uniqueness argument also works for the polynomial over $p_m^*(\gamma^1)^m, p_n^*(\gamma^1)^n$ and $p_m^*(\gamma^1)^m \otimes p_n^*(\gamma^1)^n$ over $(P^\infty)^{m+n}$, since the cohomology ring in that case is a subring of the free polynomial ring, thus without non-trivial relations.

Thus it suffices for us to find the formula when $\xi^m = \oplus_1^m \xi_i^1$ and $\eta^n = \oplus_1^n \eta_j^1$ are Whitney sums of line bundles. Then as a special case we will have the same relation for the tensor product of $p_m^*(\gamma^1)^m, p_n^*(\gamma^1)^n$ (which are Whitney sums) and then by the uniqueness argument we will know that this must correspond to the polynomial for the universal bundles. For Whitney sums of line bundles, if we assume the $m = n = 1$ case, then we see that:

$$w((\oplus_1^m \xi_i^1) \otimes (\oplus_1^n \eta_i^1)) = \prod_{i,j=1,1}^{m,n} w(\xi_i^1 \otimes \eta_j^1) = \prod_{i,j=1,1}^{m,n} (1 + w_1(\xi_i^1) + w_1(\eta_j^1))$$

Since $w(\oplus_1^m \xi_i^1) = \prod_i (1 + w_1(\xi_i^1)$, (and thus each $w_j(\oplus_1^m \xi_i^1)$ is an elementary symmetric polynomial in the $w_1(\xi_i^1)$), this certainly corresponds to the polynomial relation desired.

Finally, consider the $m = n = 1$ case. Then $p_{1,1}(x, y)$ is a linear, symmetric polynomial with $p_{1,1}(x, 0) = x$. So $p_{1,1}(x, y) = x + y$.

Problem 8-A It follows from 7.1 that the cohomology class $\mathrm{Sq}^k w_m(\xi)$ can be expressed as a polynomial in $w_1(\xi), \ldots, w_{m+k}(\xi)$. Prove Wu's explicit formula $\mathrm{Sq}^k(w_m) = \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i} w_{m+i}$.

Solution 8-A Observe that if this were true for $(\gamma^1(\mathbb{R}^\infty))^n$ then it would be true for $\gamma^n(\mathbb{R}^\infty)$ since the map $f : (\mathbb{R}P^\infty)^n \to G_n(\mathbb{R}^\infty)$ induces an algebra isomorphism on cohomology, is covered by a bundle map $(\gamma^1)^n \to \gamma^n$, and thus satisfies $f^*\mathrm{Sq}^k w_m = \mathrm{Sq}^k f^* w_m = \mathrm{Sq}^k w_m$. Then by the universality of $\gamma^n(\mathbb{R}^\infty)$ it must be true for any n-bundle. So it suffices to prove that if the formula holds for ξ, then it holds for $\gamma^1 \times \xi$. Then by induction we have the formula for $(\gamma^1)^n$.

To show this, observe that:

$$\mathrm{Sq}^k w_m(\xi \times \gamma^1) = \mathrm{Sq}^k(w_m(\xi) \times 1 + w_{m-1}(\xi) \times w_1(\gamma^1)) = \sum_{i+j=k} \mathrm{Sq}^i w_m(\xi) \times \mathrm{Sq}^j(1) + \mathrm{Sq}^i w_{m-1}(\xi) \times \mathrm{Sq}^j w_1(\gamma^1)$$

$$= \mathrm{Sq}^k w_m(\xi) \times 1 + \mathrm{Sq}^{k-1} w_m(\xi) \times (w_1(\gamma^1) \cup w_1(\gamma^1)) + \mathrm{Sq}^k w_{m-1}(\xi) \times w_1(\gamma^1)$$

$$= \sum_{i=0}^{k} \binom{k-m}{i} \big((w_{k-i}(\xi) \cup w_{m+i}(\xi)) \times 1 + (w_{k-i-1}(\xi) \cup w_{m+i-1}(\xi)) \cup (w_1(\gamma^1) \cup w_1(\gamma^1))\big)$$

$$+ \sum_{i=0}^{k} \binom{k-m+1}{i} (w_{k-i}(\xi) \cup w_{m+i-1}(\xi)) \times w_1(\gamma^1)$$

Now observe that due to the identity $\binom{k-m+1}{i} = \binom{k-m}{i-1} + \binom{k-m}{i}$ we have:

$$\sum_{i=0}^{k} \binom{k-m+1}{i} (w_{k-i}(\xi) \cup w_{m+i-1}(\xi)) \times w_1(\gamma^1) = \sum_{i=0}^{k} (\binom{k-m}{i-1} + \binom{k-m}{i}) (w_{k-i}(\xi) \cup w_{m+i-1}(\xi)) \times w_1(\gamma^1)$$

$$= \sum_{i=0}^{k} \binom{k-m}{i} (w_{k-i}(\xi) \cup w_{m+i}(\xi)) \times w_1(\gamma^1) + \sum_{i=0}^{k} \binom{k-m}{i} (w_{k-i-1}(\xi) \cup w_{m+i}(\xi)) \times w_1(\gamma^1)$$

Therefore, plugging this into the first manipulation we have:

$$\mathrm{Sq}^k w_m(\xi \times \gamma^1) = \sum_{i=0}^{k} \binom{k-m}{i} (w_{k-i}(\xi) \times 1 + w_{k-i-1}(\xi) \times w_1(\gamma^1)) \cup (w_{m+i}(\xi) \times 1 + w_{m+i-1}(\xi) \times w_1(\gamma^1))$$

$$= \sum_{i=0}^{k} \binom{k-m}{i} w_{k-i}(\xi \times \gamma^1) \cup w_{m+i}(\xi \times \gamma^1)$$

This verifies the formula in our reduced case.

Problem 8-B If $w(\xi) \neq 1$, show that the smallest $n > 0$ with $w_n(\xi) \neq 0$ is a power of 2.

Solution 8-B Assume that $2^r j$ is the smallest positive non-zero integer with $w_{2^r j}(\xi) \neq 0$. Assume for contradiction j is odd and not equal to 1, and let $k = 2^r, m = 2^r(j-1)$. Then by the Wu formula and the fact that $w_i(\xi) = 0$ for $i < m+k$ we have:

$$0 = \mathrm{Sq}^{2^r} w_{2^r(j-1)}(\xi) = \binom{2^r(2-j)}{2^r} w_{m+k}(\xi)$$

But since $j \neq 1$, $2^r(2-j)$ is odd and non-zero. Thus $\binom{2^r(2-j)}{2^r} \neq 0 \in \mathbb{Z}/2$ and this is a contradiction.

Problem 9-A Show that $\gamma^n \oplus \gamma^n$ is an orientable vector bundle with $w_{2n}(\gamma^n \oplus \gamma^n) \neq 0$ and hence $e(\gamma^n \oplus \gamma^n) \neq 0$. Show that if n is odd then $2e(\gamma^n \oplus \gamma^n) = 0$.

Solution 9-A A bundle ξ is orientable if and only if $w_1(\xi) = 0$. But $w_1(\gamma^n \oplus \gamma^n) = w_1(\gamma^n) \cup 1 + 1 \cup w_1(\gamma^n) = 2w_1(\gamma^n) = 0$. So $\gamma^n \oplus \gamma^n$ is orientable. Furthermore, $w_{2n}(\gamma^n \oplus \gamma^n) = \sum_{i+j=2n} w_i(\gamma^n) \cup w_j(\gamma^n) = w_n(\gamma^n) \cup w_n(\gamma^n)$. The latter is not 0 since $w_n(\gamma^n) \neq 0$. If $b \in H_n(G_n\mathbb{R}^\infty, \mathbb{Z}/2)$ satisfies $\langle w_n(\gamma^n), b\rangle \neq 0$, then:

$$\langle w_{2n}(\gamma^n \oplus \gamma^n), b\rangle = \langle w_n(\gamma^n) \times w_n(\gamma^n), b \times b\rangle = \pm\langle w_n(\gamma^n), b\rangle^2 \neq 0$$

Therefore $w_{2n}(\gamma^n \oplus \gamma^n) \neq 0$ and $e(\gamma^n \oplus \gamma^n) \neq 0$ because $e(\gamma^n \oplus \gamma^n) \mod 2 = w_{2n}(\gamma^n \oplus \gamma^n) \neq 0$. To see then that $2e(\gamma^n \oplus \gamma^n) = 0$ we just apply the reasoning on p. 101.

Problem 9-B Show that the restriction of the tautological bundle ξ^{2n} over $G_n\mathbb{C}^\infty$ to the subspace $G_n\mathbb{R}^\infty$ is isomorphic to $\gamma^n \oplus \gamma^n$ and hence that $e(\xi^{2n}) \neq 0$.

Solution 9-B . It's clear that we have the inclusion $G_n\mathbb{R}^\infty \subset G_n\mathbb{C}^\infty$ via the inclusion $V_o^n(\mathbb{R}^m) \subset V_o^n(\mathbb{C}^m)$ and the fact that the quotient map $q_{\mathbb{C}} : V_o^n(\mathbb{C}^m) \to G_n\mathbb{C}^m$ is compatible with the quotient $q_{\mathbb{R}} : V_o(\mathbb{R}^m) \to G_n\mathbb{R}^m$, since two real frames generate the same n-plane in $\mathbb{R}^m$ if and only if their complexifications generate the same n-plane in $\mathbb{C}^m$. Furthermore, observe that the inclusion $i : \mathbb{R}^m \subset \mathbb{C}^m$ induces an inclusion $\gamma^n(\mathbb{R}^m) \subset \xi^{2n}(\mathbb{C}^m)$ via their definitions as sub-bundles of the trivial real and complex m-bundles over $G_n(\mathbb{R}^m)$ and $G_n(\mathbb{C}^m)$ respectively. i sends the fiber of $\gamma^n(\mathbb{R}^m)$ to the real sub-space of the fiber of $\xi^{2n}(\mathbb{C}^m)$. The perpendicular bundle has fiber equal to the imaginary subspace of the fiber of $\xi^{2n}(\mathbb{C}^m)$. But we can also construct i to send the fiber of $\gamma^n(\mathbb{R}^m)$ to the imaginary subspace of $\mathbb{C}^m$ via the composition $\mathbb{R}^m \xrightarrow{i} \mathbb{C}^m \xrightarrow{\cdot\sqrt{-1}} \mathbb{C}^m$, and this induces an isomorphism between $(\gamma^n(\mathbb{R}^m))^\perp \subset \xi^{2n}(\mathbb{C}^m)$ and γ^n. So we have $\xi^{2n}(\mathbb{C}^m)|_{G_n(\mathbb{R}^m)} \simeq (\gamma^n(\mathbb{R}^m))^\perp \oplus \gamma^n(\mathbb{R}^m) \simeq \gamma^n(\mathbb{R}^m) \oplus \gamma^n(\mathbb{R}^m)$ for all m, and by taking a direct limit we get $\xi^{2n} \simeq \gamma^n \oplus \gamma^n$ as desired. Problem 9-A yields the last conclusion.

Problem 9-C Let τ be the tangent bundle of the n-sphere and let $A \subset S^n \times S^n$ be the anti-diagonal, consisting of all pairs of antipodal unit vectors. Using stereographic projection, show that the total space $E(\tau)$ of τ is canonically diffeomorphic to $S^n \times S^n - A$. Hence using excision and homotopy show that:

$$H^*(E, E_0) \simeq H^*(S^n \times S^n, S^n \times S^n - D) \simeq H^*(S^n \times S^n, A) \subset H^*(S^n \times S^n)$$

Here D is the diagonal of $S^n \times S^n$. Now suppose that n is even. Show that the Euler class $e(\tau) = \phi^{-1}(u \cup u)$ is twice a generator of $H^n(S^n, \mathbb{Z})$. As a corollary, show that τ possesses no non-trivial sub-bundles.

Solution 9-C The tangent bundle τ is isomorphic to the sub-bundle $(\nu S^n)^\perp \subset \epsilon^{n+1}$ of the trivial bundle via the embedding of S^n into $\mathbb{R}^{n+1}$. We can define a map of $\psi : S^n \times S^n - A \to \epsilon^n$ as so. Given $(p,q) \in S^n \times S^n - A$ we can write a matrix $M_{p,q} = \begin{bmatrix} \Pi_p \\ \Pi_{p+q}^\perp \end{bmatrix}$ and a vector $v_{p,q} = \begin{bmatrix} 1 \\ \Pi_{p+q}^\perp p \end{bmatrix}$. Then define the map $\psi(p,q) = (p, M_{p,q}^{-1} v_{p,q})$. This map is defined to be equal to the stereographic projection on each fiber, i.e for fixed p. Furthermore this matrix expression shows that the map is clearly smooth, injective and surjective onto the tangent bundle $\tau \subset \epsilon^{n+1}$ (easily checkable by looking at the map fiber-wise and noting that it is just the usual stereographic projection). The smooth inverse map can likewise be defined in terms of a smoothly varying matrix expression to show that the map is a diffeomorphism.

Thus we have:

$$H^*(E, E_0) \overset{\psi^*}{\simeq} H^*(S^n \times S^n - A, S^n \times S^n - (A \cup D)) \simeq$$

$$H^*(S^n \times S^n, S^n \times S^n - D) \simeq H^*(S^n \times S^n, A) \subset H^*(S^n \times S^n)$$

The first isomorphism is induced by ψ, the second is by excision of A, the third is by the fact that $(S^n \times S^n, A) \subset (S^n \times S^n, S^n \times S^n - D)$ is a homotopy equivalence ($S^n - D$ retracts onto A). [INSERT PROOF THAT $e(\tau)$ is twice a generator of $H^n(S^n, \mathbb{Z})$.

Now suppose for the sake of contradiction that $\tau = \xi \oplus \xi'$. Then $w_1(\xi) = w_1(\xi') = 0$ because S^n has no first homology. So they are orientable and $e(\tau) = \pm e(\xi) \cup e(\xi')$. But S^n has no cohomology groups of degree less than n except for $H^0(S^n, \mathbb{Z})$, so these must vanish, even though $e(\tau)$ does not. So we have a contradiction.

Problem 11-A Prove Lemma 4.3 (that is compute the mod 2 cohomology of $\mathbb{R}P^n$) by induction on n, using the Duality Theorem and the cell structure of 6.5.

Solution 11-A We will abbreviate $\mathbb{R}P^n$ as P^n. P^1 is evidently the circle and thus has cohomology $H^0(P^1) = H^1(P^1) = \mathbb{Z}/2$ and 0 for other i. Now by induction suppose that $n \geq 2$, $H^i(P^n) = \mathbb{Z}/2$ for $i \leq n$ and 0 for other i. Then we have the long-exact sequence of the pair (P^{n+1}, P^n):

$$\cdots \to H^i(P^{n+1}, P^n) \to H^i(P^{n+1}) \to H^i(P^n) \to H^{i+1}(P^{n+1}, P^n) \to \ldots$$

Using the cell structure of 6.5 we know that the relative cochain groups $C^i(P^{n+1}, P^n)$ is 0 for $i < n$. In particular, $\frac{n+1}{2} < n$ so $H^i(P^{n+1}) \simeq H^i(P^n) = \mathbb{Z}/2$ for such i. But then by Poincare duality we can conclude that $H^i((P^{n+1}) = \mathbb{Z}/2$ for all $i \leq n+1$ and 0 otherwise (by dimensionality).

Problem 11-B (More Poincare Duality) For M compact, using field coefficients, show that $u''/ : H_{n-k} \to H^k(M)$ is an isomorphism. Using the cap product operation of Appendix A, show that the invest isomorphism is given by $\cup\mu : H^k(M) \to H_{n-k}(M)$ multiplied by the sign $(-1)^{kn}$. Note: I believe that M/S means $(-1)^{k(n-k)}$ here.

Solution 11-B Assume that all (co)homology is over a field. We utilize the formula $u'' = \sum_i b_i \times b_i^\#$ for some homogeneous basis b_i of $H^*(M)$. Assume $b_i \in H^k(M)$. We know that $H^{n-k}(M)$ is canonically isomorphic to the dual of $H_{n-k}(M)$. Thus we can take $(b_i^\#)^v$ to be the dual basis element of $b_i^\#$ in $H_{n-k}(M)$. By the linearity of the map $\beta \mapsto u''/\beta$ and dimension counting from 11.10, it suffices here to show that a basis element $(b_i^\#)^v \in H_{n-k}(M)$ map to independent elements of $H^n(M)$ under $u''/$. To see that, observe:

$$u''/(b_i^\#)^v = \sum_j b_j \langle b_j^\#, (b_i^\#)^v \rangle = \sum_j b_j \delta_{ij} = b_i \in H^n(M)$$

Now we verify the inverse map. By the reasoning above, it suffices to check that $(-1)^{(n-k)k} b_i \cap \mu = (b_i^\#)^v$, or equivalently that $(-1)^{k(n-k)} \langle b_j^\#, b_i \cap \mu \rangle = \delta_{ij}$. But observe that by the definition of the cap product we have $\langle a, b \cap \mu \rangle = \langle a \cup b, \mu \rangle$. So $(-1)^{k(n-k)} \langle b_j^\#, b_i \cap \mu \rangle = \langle b_i \cup b_j^\#, \mu \rangle$. The last expression is δ_{ij} by $b_j^\#$'s definition.

Problem 11-C Let $M = M^n$ and $A = A^p$ be compact oriented manifolds with smooth embedding $i : M \to A$. Let $k = p - n$. Show that Poincare duality isomorphism $\cap_{\mu_A} : H^k(A) \to H_n(A)$ maps the cohomology class $u'|A$ "dual" to M to the homology class $(-1)^{nk} i_*(\mu_M)$.

Solution 11-C

Problem 11-D Prove that all Stiefel-Whitney numbers of a 3-manifold are 0.

Solution 11-D Let $w[\tau] = Sq(v)$ be the total Stiefel-Whitney class of the tangent bundle τ of a 3-fold M. The only partitions of 3 are $(1,1,1)$, $(1,2)$ and (3). So the only Stiefel-Whitney numbers are $\langle w_1^3[\tau], \mu \rangle, \langle w_1[\tau] w_2[\tau], \mu \rangle$ and $\langle w_3[\tau], \mu \rangle$. Now observe that $v_2, v_3 = 0$ since $3 > 0, 2 > 1$. Thus using the Wu formula, $w_3[\tau] = 0, w_2[\tau] = v_1 \cup v_1$ and $w_1[\tau] = v_1$. Therefore it suffices to prove that $\langle v_1^3, \mu \rangle = \langle Sq^1(v_1 \cup v_1), \mu \rangle = 0$. But $\mathrm{Sq}^1(v_1 \cup v_1) = \mathrm{Sq}^0(v_1) \cup \mathrm{Sq}^1(v_1) + \mathrm{Sq}^1(v_1) \cup \mathrm{Sq}^0(v_1) = 2(v_1 \cup v_1 \cup v_1) = 0$. So the last expression vanishes.

Problem 11-E Prove the following version of Wu's formula. Let $\bar{Sq} : H^*(M) \to H^*(M)$ be the inverse of the ring automorphism Sq. Prove that $\bar{w}$ is determined by the formula $\langle \bar{Sq}(x), \mu \rangle = \langle \bar{w} \cup x, \mu \rangle$. Show that $\bar{w}_n = 0$. If n is not a power of 2, show that $\bar{w}_{n-1} = 0$.

Solution 11-E Observe that $\bar{w} \cup Sq(v) = \bar{w} \cup w = 1$ can be written as $\bar{Sq}(\bar{w}) \cup v = 1$. Thus if we set $y = Sq(x)$, we have:

$$\langle \bar{Sq}(y), \mu \rangle = \langle \bar{Sq} Sq(x), \mu \rangle = \langle x, \mu \rangle = \langle v \cup \bar{Sq}(w) \cup x, \mu \rangle = \langle Sq(\bar{Sq}(w) \cup x), \mu \rangle = \langle \bar{w} \cup y, \mu \rangle$$

To see that $\bar{w}_n = 0$, observe that for any $x \in H^0(X)$ we have $Sq(x) = Sq^0(x) = x$, and thus $\bar{Sq}(x) = x$. Therefore $0 = \langle x, \mu \rangle = \langle \bar{Sq}(x), \mu \rangle = \langle \bar{w}_n \cup x, \mu \rangle$. But now note that μ breaks into a direct sum of fundamental classes of X's connected components X_i, each of which generated $H^n(X_i)$. Thus $\mu = \oplus_1^k \mu_i$ and $H^0(X)$ likewise breaks into a direct sum of $H^0(X_i) \simeq \mathbb{Z}/2$. If $\bar{w}_n$ were not 0, then in its direct sum decomposition $\bar{w}_n = \oplus_i \bar{w}_n^i \in H^n(X_i)$ we would have $\bar{w}_n^i = \mu_i^v$ (the dual of μ_i^v) for some i. Then if we took the element $1_i \in H^0(X_i) \subset H^0(X)$, we see that $\langle \bar{w}_n \cup 1_i, \mu \rangle = \langle \mu_i, \mu_i \rangle \neq 0$, a contradiction. So $\bar{w}_n = 0$.

Finally, if n is even then $\bar{w}_{n-1}$

Problem 11-F Defining Steenrod operations $Sq^i : H_k(X) \to H_{k-i}(X)$ on mod 2 cohomology by the identity $\langle x, Sq^i(\beta)\rangle = \langle \bar{Sq}^i(x), \beta\rangle$, show that $Sq(a \cap \beta) = Sq(a) \cap Sq(\beta)$ and that $Sq(u''/\beta) = Sq(u'')/Sq(\beta)$. Prove the formulae $Sq(\mu) = \bar{w} \cap \mu$ and $\bar{Sq}(\mu) = v \cap \mu$.

Solution 11-F For the first identity, observe the following manipulation:

$$\langle x, Sq(a) \cap Sq(\beta)\rangle = \langle x \cup Sq(a), Sq(\beta)\rangle = \langle \bar{Sq}(x \cup Sq(a)), \beta\rangle =$$

$$\langle \bar{Sq}(x) \cup a, \beta\rangle = \langle \bar{Sq}(x), a \cap \beta\rangle = \langle x, Sq(a \cap \beta)\rangle$$

Since this manipulation is valid for any $x \in H^*(X)$, we must have that $Sq(a) \cap Sq(\beta) = Sq(a \cap \beta)$. For the second identity, observe that we can use the formula $u'' = \sum_i (-1)^{\dim b_i} b_i \times b_i^{\#}$ as so:

$$Sq(u''/\beta) = \sum_i (-1)^{\dim b_i} Sq(b_i)\langle b_i^{\#}, \beta\rangle = \sum_i (-1)^{\dim b_i} Sq(b_i)\langle Sq(b_i^{\#}), Sq(\beta)\rangle$$

$$= \sum_i (-1)^{\dim b_i} [Sq(b_i) \times Sq(b_i^{\#})]/Sq(\beta)\rangle = Sq(u'')/Sq(\beta)$$

The last formulae $Sq(\mu) = \bar{w} \cap \mu$ and $\bar{Sq}(\mu) = v \cap \mu$ follow immediately from the definition of v on p. 132 and 11-E.

Problem 12-A Prove that a vector bundle ξ over a CW-complex is orientable if and only if $w_1(\xi) = 0$.

Solution 12-A The forward direction is done on p. 146. Conversely, an orientation on ξ is evidently equivalent to a global section of the $\mathbb{Z}/2$ bundle of connected components of $V^n\xi$ (two n-frames are in the same component if they are related by an orientation preserving transformation), which is just a global section of $\{\pi_0 V^n \xi\}$. But the existence of such a section is precisely measured by the vanishing of all of the obstruction classes $H^i(B, \{\pi_{i-1}(E)\})$ where $E \to B$ is the $\mathbb{Z}/2$ fiber bundle of components of $V^n\xi$. The only non-zero obstruction is just $H^1(B, \{\pi_0 V^n \xi\}) = \mathbf{o}_1(\xi) = w_1(\xi) = 0$. The fact that $w_1 = \mathbf{o}_1$ follows from the discussion on p.143.

Problem 12-B Using the Wu formula 11.14 and the fact that $\pi_2 V_2(\mathbb{R}^3) \simeq \pi_2 SO(3) = 0$, prove Stiefel's theorem that every compact orientable 3-manifold is parallelizable.

Solution 12-B Let M be an orientable 3-fold with tangent bundle τ. Via the computations of $w(\tau)$ given in Solution 11-D and the assumption that $w_1(\tau) = 0$ (via the orientability), we know that $w(\tau) = 1$. Thus $\mathbf{o}_i(\tau) = 0$ by the results of p. 143, since everything is determined by $w_i(\tau)$ in this case. Now observe that τ is parallelizable if and only if we can find a global section of the Stiefel bundle $V_3\tau$. This occurs if and only if all of the obstructions $\mathbf{o}_i(V_3\tau) \in H^i(M; \{\pi_{i-1}V_3\tau\})$ all vanish.

Now observe that $V_3\mathbb{R}^3$ deformation retracts to the orthogonal Stiefel manifold $V_3^o\mathbb{R}^3$ via a Graham-Schmidt process. This, in turn, is simply a $\mathbb{Z}/2$ bundle over $V_2^o\mathbb{R}^3$ (since every orthogonal $n-1$-frame has only two possible extensions to an n-frame by adding the unique (up to sign) perpendicular unit vector). $V_2^o\mathbb{R}^3$ is, in turn, a deformation retract of $V_2\mathbb{R}^3$. This implies that $\pi_i V_3\tau) \simeq \pi_i V_2\tau$ for $i > 0$ via the fiber-wise inclusion taking a 2-frame to the unique oriented 3-frame extension of the 2-frame by a unit vector. Thus the obstructions $\mathbf{o}_i(V_3\tau) = \mathbf{o}_i(V_2\tau)$ via this isomorphism for $i > 1$.

With this information, we can show that all of the obstructions vanish. First, via the discussion on p. 140 we know that we can start with $i = 1$. By our previous discussion, the obstruction to extending a section over the 0-cells in M to a section over the 1-cells is $\mathbf{o}_1(V_3\tau) = \mathbf{o}_1(\tau) = 0 \in H^1(M; \{\pi_0 V_3\tau\})$. By the homotopy group isomorphism given above, we also have $\mathbf{o}_2(V_3\tau) = \mathbf{o}_2(V_2\tau) = \mathbf{o}_2(\tau) = w_2(\tau) = 0$, via the string of isomorphisms and arguments given above. Finally, $H^3(M; \{\pi_2 V_3\tau\}) = H^3(M; 0)$ since $\pi_2 V_3\tau = \pi_2 V_2\tau = \pi_2 SO(3) = 0$. The higher obstructions vanish by the dimension of M. Thus all obstructions vanish, a global section exists and τ is trivial.

Problem 12-C Use Corollary 12.3 to give another proof that $H^*(P^n; \mathbb{Z}/2)$ is as described in Section 4.3.

Solution 12-C We have a long exact sequence $\cdots \to H^{i-1}(P^n) \to H^i(P^n) \to H^i(S^n) \to H^i(P^n) \to H^{i+1}(P^n) \to \ldots$. The initial part of the sequence reads $0 \to H^0(P^n) \to H^0(S^n) \to H^0(P^n) \to H^1(P^n) \to \ldots$. Since $H^0(S^n) \simeq H^0(P^n) \simeq \mathbb{Z}/2$, since both spaces are connected, and the second map in $0 \to H^0(P^n) \to H^0(S^n)$ is injective, we can conclude that the first map in $H^0(S^n) \to H^0(P^n) \to H^1(P^n) \to \ldots$ is 0 and thus that $H^0(P^n) \simeq H^1(P^n)$. By the same reasoning, and using the fact that $H^i(S^n) = 0$ for all $0 < i < n$, we can conclude that $H^i(P^n) \simeq H^{i+1}(P^n)$ for $i + 1 < n$. Furthermore $H^i(P^n) = 0$ for $i > n$ obviously, by dimensionality. To get $H^n(P^n)$ just observe the last part of the exact sequence $H^{n-1}(P^{n-1}) \to H^n(P^n) \to H^n(S^n) \to H^n(P^n) \to H^{n+1}(P^n) = 0$. Since $H^n(S^n) = \mathbb{Z}/2$, and exactness at $H^n(P^n)$ implies $H^n(S^n) \to H^n(P^n)$ is surjective, $H^n(P^n) = \mathbb{Z}/2$ as well. Exact sequences are crazy!

Problem 12-D Show that $\tilde{G}_n(\mathbb{R}^{n+k})$ is a smooth, compact, orientable manifold of dimension nk. Show that the correspondence which maps a plane with oriented basis $b_1, \ldots, b_n$ to $\wedge_i b_i / |\wedge_i b_i|)$ embeds $\tilde{G}_n(\mathbb{R}^{n+k})$ smoothly in $\wedge^n \mathbb{R}^{n+k})$.

Solution 12-D We can boot strap on the fact that we already know that $G_n(\mathbb{R}^{n+k})$ is a manifold. In particular, we have the exact sequence of Lie groups $0 \to SO_n \to O_n \to \mathbb{Z}/2 \to 0$. Via the quotient space construction of $G_k(\mathbb{R}^{n+k})$ and $\tilde{G}_n(\mathbb{R}^{n+k})$, we thus have the quotient map $V_k^o\mathbb{R}^{n+k} \xrightarrow{q} G_n(\mathbb{R}^{n+k})$ factorizing as $V_k^o\mathbb{R}^{n+k} \xrightarrow{q_o} \tilde{G}_n(\mathbb{R}^{n+k}) \xrightarrow{\tilde{q}} G_n(\mathbb{R}^{n+k})$ where q_o is the quotient by the SO_n group action and $\tilde{q}$ is the quotient by the residual $\mathbb{Z}/2$ action.

The $\tilde{q}$ map is given by a free finite group action of $\mathbb{Z}/2$, so the map is a topological double cover map. Hausdorff-ness is preserved; given 2 points $x \neq y \in \tilde{G}_n(\mathbb{R}^{n+k})$, if $\tilde{q}(x) = \tilde{q}(y)$, then any simply connected open set U of $q(x)$ can be lifted to a disconnected open set $\tilde{q}^{-1}(U)$ with two components containing x and

y respectively. If $q(x) \neq q(y)$, it suffices to take disjoint neighborhoods of $q(x)$ and $q(y)$ then lift them. Compactness of $\tilde{G}_n(\mathbb{R}^{n+k})$ is given by the fact that it is the quotient of a compact space. It's smooth structure can be given by taking any atlas of simply connected opens U_α on $G_n(\mathbb{R}^{n+k})$ and lifting it to $q^{-1}U_\alpha$ (and splitting each inverse image into its two components). Then each open in the resulting atlas is homeomorphic to an open in $\mathbb{R}^{nk}$ and the transition functions are smooth since any intersection of two such neighborhoods is homeomorphic to its image intersection in $G_n(\mathbb{R}^{n+k})$. The dimension of $\tilde{G}_n(\mathbb{R}^{n_k})$ then follows from the local homeomorphism property of the double cover.

To see orientability, observe that via the pullback through $\tilde{q}$ we have the isomorphism $\tau \simeq \mathrm{Hom}(\tilde{\gamma}^n, (\tilde{\gamma}^n)^\perp)$ $\simeq \tilde{\gamma}^n \otimes (\tilde{\gamma}^n)^\perp$. Via the tensor product formula for $w(\tilde{\gamma}^n \otimes (\tilde{\gamma}^n)^\perp)$, the fact that $w((\tilde{\gamma}^n)^\perp) = \bar{w}(\tilde{\gamma}^n)$, the fact that $\bar{w}_1 = w_1$ in general and the fact that $w_1(\tilde{\gamma}^n) = 0$ by the results of this chapter, we can conclude that $w(\tilde{\gamma}^n \otimes (\tilde{\gamma}^n)^\perp)$ is a polynomial in cohomology classes of degree 2 and higher, so it can't have a non-zero w_1 term, so $G_n(\mathbb{R}^{n+k})$ is orientable. We could also just argue directly from the definition of $\tilde{G}_n(\mathbb{R}^{n+k})$ to give it a natural orientation, but this is kind of a neat argument.

At last, to see that the map $\tilde{G}_n(\mathbb{R}^{n+k}) \to \wedge^n \mathbb{R}^{n+k}$ defined above is indeed a smooth embedding, we observe the following. First, we observe that it is smooth n $V_n\mathbb{R}^{n+k}$ since it is a composition of smooth well-defined functions. The wedge product is polynomial in b_i and the norm is smooth away from 0. Next observe that this map is equivariant under the $GL_n^o(n)$ (oriented general linear) action. Indeed, an oriented linear recombination M of the b_i changed both the top and bottom of the quotient expression by a multiplicative factor of $|\det M| = \det M$ (since M is oriented). Finally, injectivity is provided by the fact that if $\wedge_i b_i = \lambda \wedge_i c_i$ for non-zero λ then $c_j \wedge (\wedge_i b_i) = 0$ for all j, so all c_j are in the span of b_i implying that their span is the same plane. Furthermore, if $\lambda > 0$ then the bases b_i and c_i are related by an orientation preserving transformation, because otherwise we would have $\wedge_i b_i = \det M \wedge_i c_i$ for $\det M < 0$.

Problem 13-A Show that a complex structure $J : E(\xi) \to E(\xi)$ always satisfies the local triviality condition.

Solution 13-A Observe that a $2n \times 2n$ matrix J satisfying $J^2 = -Id$ is conjugate to the usual $J_0 = \begin{pmatrix} 0 & Id \\ -Id & 0 \end{pmatrix}$ via a real change of coordinates. Indeed, since the characteristic polynomial of both J and J_0 are both real with eigenvalues i and $-i$ (since $J^2 = -1$ implies that J has eigenvalues $\pm i$) the eigenvalues of both must be $\pm i$ in conjugate pairs (thus, n i's and n $-i$'s). If you then examine the complex Jordan normal form of both matrices, you see that the condition $J^2 = J_0^2 = -Id$ implies that the Jordan normal forms are fully diagonal, and thus complex J is diagonalizable as is J_0, and they have the same Jordan normal form. Thus they are complex conjugate, and therefore real conjugate (since conjugacy between two real matrices is the same taken over $\mathbb{R}$ or $\mathbb{C}$).

Thus there is a change of coordinates sending J to J_0. Now suppose that we are given a complex structure J on a $2n$-plane bundle ξ. At any point p we can pick a neighborhood U with $\xi|U \simeq \epsilon^{2n}$ and $E|U \simeq U \times \mathbb{R}^{2n}$ i.e with trivial restriction. Thus near the point p we can treat J as a field of matrices $J_p : \mathbb{R}^{2n} \to \mathbb{R}^{2n}$. We know that at each fiber ξ_p we can pick a change of coordinates T_p so that $T_p J_p T_p^{-1} = J_0$. Furthermore, such a choice can be made smoothly, since the 2d "eigenspaces" on which J_p takes the form

$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ vary smoothly with p. Such a smooth field of coordinate transformations T_p thus yields our desired trivialization.

Problem 13-B If M is a complex manifold, show that DM is complex. Similar, show that if $f : M \to N$ is holomorphic then so is $Df : DM \to DN$.

Solution 13-B M is complex with complex structure J if there exists a local trivialization $\phi : U \to V \subset \mathbb{C}^n$ around any point p such that $J_0 D\phi = D\phi J$. Since $D(DM) = H(DM) \oplus V(DM)$, a horizontal and vertical component (the kernel and coimage of the bundle projection $D(DM) \to \pi^* DM$). There is a natural bundle isomorphism $v : \pi^* DM \to V(DM)$ given by $v_Y(X)f = \frac{d}{dt}_{t=0} f(x, Y + tX)$ and a natural isomorphism $H(DM) \to \pi^* DM$ given by the inverse of $\pi^* : H(DM) \to DM$. Thus $D(DM) \simeq \pi^* DM \oplus \pi^* DM$ via the two natural isomorphisms listed above and we can define a complex structure on J as $\tilde{J} = (\pi^*)^{-1} J \pi^* \oplus v J v^{-1}$. In coordinates $(x_1, \ldots, x_{2n}, \xi_1, \ldots, \xi_{2n})$ on a double tangent fiber this is just expressing $\tilde{J}$ as $J \oplus J$.

Now, in the same coordinates we see that the map $D\phi : DU \to DV$ splits as $(p, \nu) \mapsto (\phi(p), D\phi_p \nu)$ and has differential $D(D\phi) = \begin{pmatrix} D\phi_p & 0 \\ D^2\phi_p \nu & D\phi_p \end{pmatrix}$. We can verify $\tilde{J}_0 D(D\phi) = D(D\phi) J_0$ by observing:

$$\begin{pmatrix} J_0 & 0 \\ 0 & J_0 \end{pmatrix} \begin{pmatrix} D\phi_p & 0 \\ D^2\phi_p \nu & D\phi_p \end{pmatrix} = \begin{pmatrix} J_0 D\phi_p & 0 \\ J_0 D^2\phi_p \nu & J_0 D\phi_p \end{pmatrix} = \begin{pmatrix} D\phi_p & 0 \\ D^2\phi_p \nu & D\phi_p \end{pmatrix} \begin{pmatrix} J & 0 \\ 0 & J \end{pmatrix}$$

Problem 13-C If M is a compact complex manifold, show that every holomorphic map $f : M \to \mathbb{C}$ is constant on connected components.

Solution 13-C On each component M_i of M, f takes a maximum modulus, say $f(p_i)$ at some point p_i. We show that the set of points in M_i with $f(p) = f(p_i)$ is open, closed, and non-empty. It is non-empty by construction. It is closed because f is continuous. It is open by the maximum modulus principle. Any p with $f(p) = f(p_i)$ in M_i has a neighborhood U biholomorphic to $U \subset \mathbb{C}$. Then f is a holomorphic function obtaining a maximum in the interior of its domain, making it constant. An open and closed component of a connected manifold is the whole thing. So f is constant on each M_i. This can be seen as a consequence of de Rham cohomology as well.

Problem 13-D Show that the projective space $\mathbb{C}P^n$ can be given the structure of a complex manifold. More generally show that the space $G_k(\mathbb{C}^n)$ is a complex manifold of complex dimension $k(n-k)$.

Solution 13-D It suffices to illustrate coordinate charts with holomorphic transition maps. This is a very standard exercise in algebraic geometry. In particular, you can put projective coordinates $[x_0, \ldots, x_n]$ on $\mathbb{C}P^n$ (where $[x_0, \ldots, x_n] = \lambda[x_0, \ldots, x_n]$ for all $\lambda \neq 0$) then use as charts the open sets U_i with $[x_0, \ldots, x_n]$ all satisfying $x_i \neq 0$ and chart maps $[x_0, \ldots, x_n] \mapsto \frac{1}{x_i}(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. The transition maps are easily seen to be non-singular rational functions, so they are holomorphic.

More generally, one can use exactly the same approach as in Problem 5-A (see the solution above). The same coordinate patches and transition functions work again when defined over $\mathbb{C}$ because all of the smooth transition functions in the end were rational and non-singular, so their complex counterparts are holomorphic. The complex dimension counting also gives $k(n-k)$ as desired because the matrix coordinate patches give a local bihilomorphism to the space of $k \times (n-k)$ matrices over $\mathbb{C}$. Hausdorffness and compactness come directly from the quotient description of the Grassmanian as $V_k\mathbb{C}^n/GL_k(\mathbb{C})$ via the same arguments as in Section 5.

Problem 13-E Show that γ_n^1 does not possess any non-zero global holomorphic sections. Show however that the dual $\mathrm{Hom}_{\mathbb{C}}(\gamma_n^1, \mathbb{C})$ contains atleast $n+1$ independent holomorphic sections.

Solution 13-E We characterize the tautological bundle γ_n^1 as the sub-bundle of ϵ^n via:

$$\{([x_0,\ldots,x_n],(v_0,\ldots,v_n)) \in \mathbb{C}P^n \times \mathbb{C}^{n+1} | (v_0,\ldots,v_n) = \lambda(x_0,\ldots,x_n), \lambda \in \mathbb{C}\}$$

Local trivializations are given over the U_i charts mentioned above via:

$$\phi_i : \gamma_n^1|_{U_i} \to V_i \times \mathbb{C}; ([x_0,\ldots,x_n],(v_0,\ldots,v_n)) \mapsto (\frac{1}{x_i}(x_0,\ldots,x_{i-1},x_{i+1},\ldots,x_n), v_i)$$

Note that ϕ_i^{-i} is given by:

$$((\hat{x}_0,\ldots,\hat{x}_{i-1},\hat{x}_{i+1},\ldots,\hat{x}_n),\hat{v}_i) \mapsto ([\hat{x}_0,\ldots,\hat{x}_{i-1},1,\hat{x}_{i+1},\ldots,\hat{x}_n], \hat{v}_i(\hat{x}_0,\ldots,\hat{x}_{i-1},1,\hat{x}_{i+1},\ldots,\hat{x}_n))$$

Therefore the transition functions are:

$$((\hat{x}_0,\ldots,\hat{x}_{i-1},\hat{x}_{i+1},\ldots,\hat{x}_n),\hat{v}_i) \mapsto (\frac{1}{\hat{x}_j}(\hat{x}_0,\ldots,\hat{x}_{i-1},1,\hat{x}_{i+1},\ldots,\hat{x}_{j-1},\hat{x}_{j+1},\ldots,\hat{x}_n), \hat{v}_i\hat{x}_j)$$

However, in order for a section $v : \mathbb{C}P^n \to \gamma_n^1$ to be holomorphic, it must be non-singular in each coordinate patch. If we assume that $\hat{v}_i$ is the a holomorphic function of $\hat{x}_k$ for $k \neq i$, then under the transition function change of coordinates $\phi_j\phi_i^{-1}$, $\hat{v}_i$ becomes $\frac{1}{\hat{x}_j}\hat{v}_i(\hat{x}_0/\hat{x}_j,\ldots,1/\hat{x}_j,\ldots,\hat{x}_n/\hat{x}_j)$. In particular, such a function is not holomorphic if $\hat{v}_i$ is holomorphic (which is obvious from a Taylor expansion).

The dual picture is much less bad. By duality, the transition functions for $\mathrm{Hom}(\gamma_n^1,\mathbb{C})$ must be:

$$((\hat{x}_0,\ldots,\hat{x}_{i-1},\hat{x}_{i+1},\ldots,\hat{x}_n),\hat{v}_i) \mapsto (\frac{1}{\hat{x}_j}(\hat{x}_0,\ldots,\hat{x}_{i-1},1,\hat{x}_{i+1},\ldots,\hat{x}_{j-1},\hat{x}_{j+1},\ldots,\hat{x}_n), \frac{\hat{v}_i}{\hat{x}_j})$$

So in particular, a holomorphic $\hat{v}_i$ transforms to $\hat{x}_j\hat{v}_i(\hat{x}_0/\hat{x}_j,\ldots,1/\hat{x}_j,\ldots,\hat{x}_{j-1}/\hat{x}_j,\hat{x}_{j+1}/\hat{x}_j,\ldots,\hat{x}_n/\hat{x}_j)$. Thus we can define sections s_i as 1 on U_i and $\hat{x}_i$ everywhere else. It's easy to check that this definition is consistent using the transition law given above. Each of these $n+1$ defined sections is holomorphic on the charts and independent since they are all polynomially independent on each chart.

Problem 13-F Show that the complexification $\mathrm{Hom}(\tau_M, \mathbb{R}) \otimes \mathbb{C} \simeq \mathrm{Hom}_{\mathbb{R}}(\tau_M, \mathbb{C})$ is a complex 2n-plane bundle which splits canonically as a Whitney sum $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C}) \oplus \overline{\mathrm{Hom}}_{\mathbb{C}}(\tau_M, \mathbb{C})$. If U is an open set with coordinate functions $z_1, \ldots, z_n : U \to \mathbb{C}$ show that the total differentials dz_i and $d\bar{z}_i$ for bases for $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C})$ and $\overline{\mathrm{Hom}}_{\mathbb{C}}(\tau_M, \mathbb{C})$ respectively. Writing $df = \partial f + \bar{\partial} f = \sum_i \frac{\partial f}{\partial z_i} dz_i + \frac{\partial f}{\partial \bar{z}_i} d\bar{z}_i$, show that $\frac{\partial f}{\partial \bar{z}_j} = \frac{1}{2}(\frac{\partial f}{\partial x_j} + i\frac{\partial f}{\partial y_j})$. Show that the Cauchy Riemann equations are $\frac{\partial f}{\partial \bar{z}_j} = 0$.

Solution 13-F Since $J : \tau_M \to \tau_M$ obeys $J^2 = -Id$, J induces a bundle map $J_* : \mathrm{Hom}_{\mathbb{R}}(\tau_M, \mathbb{C}) \to \mathrm{Hom}_{\mathbb{R}}(\tau_M, \mathbb{C})$ which likewise has $J_*^2 = -Id$. Since each fiber of $\mathrm{Hom}_{\mathbb{R}}(\tau_M, \mathbb{C})$ is a complex vector-space, the discussion given in 13-A shows that it must split into a Whitney sum of a $2n$-dimensional i-eigenspace $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C})$ and a $2n$-dimensional $-i$-eigenspace $\overline{\mathrm{Hom}}_{\mathbb{C}}(\tau_M, \mathbb{C})$. The covectors in these spaces satisfy $f(Jv) = if(v)$ and $\bar{f}(Jv) = -if(v)$ respectively. Furthermore, they have natural complex structures induced by J, that is $J_* f = fJ = if$ and $J_* \bar{f} = -\bar{f}J = i\bar{f}$ which are compatible with multiplication by i. So $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C})$ and $\bar{\mathrm{Hom}}_{\mathbb{C}}(\tau_M, \mathbb{C})$ are n dimensional complex vector bundles with structure J_* (which are conjugate).

If z_i are coordinate functions $z : U \to \mathbb{C}^n$, then by definition we have $iDz = DzJ$ which coordinate-wise reads $idz_i = dz_i J$. $dz_i \in \mathrm{Hom}_{\mathbb{R}}(\tau_M, \mathbb{C})$ automatically, so these are local sections of $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C})$. Since Dz is non-degenertae, they are independent. So dz_i is a basis of $\mathrm{Hom}_{\mathbb{C}}(\tau_M, \mathbb{C})$ at each fiber. $\bar{z}$ is a holomorphic function to $\bar{\mathbb{C}}$ so $-iD = DzJ$ and by all the same arguments $d\bar{z}_i$ are a basis of $\overline{\mathrm{Hom}}_{\mathbb{C}}(\tau_M, \mathbb{C})$.

For the last part, observe that $\frac{\partial f}{\partial z_i}(dx_i + idy_i) + \frac{\partial f}{\partial \bar{z}_i}(dx_i - idy_i) = \frac{\partial f}{\partial x_i} dx_i + \frac{\partial f}{\partial y_i} dy_i$ implies $\frac{\partial f}{\partial x_i} = \frac{\partial f}{\partial z_i} + \frac{\partial f}{\partial \bar{z}_i}$ and $\frac{\partial f}{\partial y_i} = i\frac{\partial f}{\partial z_i} - i\frac{\partial f}{\partial \bar{z}_i}$. Solving this linear system gives the above formula for $\frac{\partial f}{\partial \bar{z}_i}$. Plugging in $f = u + iv$ we see that $\frac{\partial f}{\partial \bar{z}_i} = 0$ is $\frac{\partial u}{\partial x_j} + i\frac{\partial v}{\partial x_j} + i\frac{\partial u}{\partial y_j} - \frac{\partial v}{\partial y_j} = 0$ which is precisely the Cauchy Riemann equations if we separate the real and imaginary parts.

Problem 13-G Show that the complex vector space spanned by the differential operators $\frac{\partial}{\partial z_i}$ at z is canonically isomorphic to the tangent space DU_z.

Solution 13-G

Problem 14-A Use Lemma 14.9 to give another proof that the tangent bundle of $\mathbb{C}P^1$ is not isomorphic to its conjugate bundle.

Solution 14-A An isomorphism $\phi : \tau \to \bar{\tau}$ would yield $-c_1(\tau) = c_1(\bar{\tau}) = \bar{\phi}^* c_1(\bar{\tau}) = c_1(\tau)$ (since the bundle map would be the identity on the underlying manifold). But since c_1 is an integer cohomology class in $\mathbb{C}P^1$ (which has $H^*(\mathbb{C}P^1) \simeq \mathbb{Z}[c_1]$) this isn't possible unless $c_1(\tau) = 0$. However, we know that $c_1(\tau) = e(\tau)$ and $\langle e(\tau), [\mathbb{C}P^1] \rangle = \chi(\mathbb{C}P^1) \neq 0$. So indeed this isn't possible.

Problem 14-B Using Property 9.5, prove inductively that the coefficient homomorphism $H^i(B, \mathbb{Z}) \to H^i(B, \mathbb{Z}/2)$ maps the total Chern class $c(\omega)$ to the total Stiefel Whitney class $w(\omega_{\mathbb{R}})$. In particular show that the odd Stiefel-Whitney classes of $\omega_{\mathbb{R}}$.

Solution 14-B We proceed by induction. This is clearly true for $n = 1$ (a complex line bundle) by 9.5 and the fact that $c_1(\xi) = e(\xi)$ and $c_0(\xi) = 1$. Now induction. If ξ is a complex n-bundle, then by the same argument as in the line case, $c_n(\xi) \mod 2 = e(\xi)$. Now consider the map $\bar{\rho} : E_0 \to B$. This is covered by a bundle map $\rho : \xi_0 \oplus \epsilon^1 = \bar{\rho}^*\xi \to \xi$. Indeed, since ξ_0 is composed of pairs (x, v) with $x \cdot v = 0 \in \xi_{\pi(x)}$ (with a Hermitian metric), its complement $(\xi_0)^\perp \subset \pi^*\xi$ can be identified as the line bundle $(x, \lambda x)$. This bundle clearly admits a global section, which is (x, x), so it is trivial.

Thus we have $\bar{\rho}^* w_i(\xi) = w_i(\xi_0 \oplus \epsilon^1) = w_i(\xi_0)$ and by the construction of the Chern classes $\bar{\rho}^* c_i(\xi) = c_i(\xi_0)$ for $i < n$. Furthermore, the map $\bar{\rho}^* : H^i(B) \to H^i(E_0)$ is an isomorphism for $i < n$ (with either $\mathbb{Z}$ or $\mathbb{Z}/2$) using, for instance, the Gysin sequence. So we have by our induction hypothesis:

$$w_{2i}(\xi) = (\bar{\rho}^*)^{-1} w_{2i}(\xi_0) = (\bar{\rho}^*)^{-1} c_i(\xi_0) \mod 2 = c_i(\xi) \mod 2$$

For odd $i < n$, the Stiefel Whitney classes $w_i(\xi_0)$ of ξ_0 are 0, thus so are $w_i(\xi)$. Thus $w(\xi) = c(\xi) \mod 2$ for an n-bundle and our induction is complete.

Problem 14-C Let $V_{n-q}(\mathbb{C}^n)$ denote the complex Stiefel manifold consisting of all complex $(n-q)$-frames in $\mathbb{C}^n$, where $0 \le q < n$. According to [Steerod], this manifold is $2q$-connected and $\pi_{2q+1} V_{n-q}(\mathbb{C}^n) \simeq \mathbb{Z}$. Given a complex n-plane bundle ω over a CW-complex B with typical fiver F, construct an associated bundle $V_{n-q}(\omega)$ over B with typical fiber $V_{n-q}(F)$. Show that the primary obstruction to the existence of a cross-section of $V_{n-q}(\omega)$ is a cohomology class in $H^{2q+2}(B, \{\pi_{2q+1} V_{n-q}(F))$ which can be identified with the Chern class $c_{q+1}(\omega)$.

Solution 14-C It's very surprising to me that Milnor is asking the reader to prove this given that he hasn't really discussed obstruction theory enough. I'll get back to this when I look at Steenrod.

Problem 14-D In analogy with Section 6, construct a cell subdivision for the complex Grassman manifold $G_n(\mathbb{C}^\infty)$ with one cell of dimension $2k$ corresponding to each partition of k integers into at most n integers. Show that the Chern class $c_k(\gamma^n)$ corresponds to the coccyx which assigns ± 1 to the Schubert cell indexed by the partition $1, \ldots, 1$ of k and zero to all the other cells.

Solution 14-D Let's start by defining these cells. Consider $\mathbb{C}^\infty$ with the natural flag $\mathbb{C}^0 \subset \mathbb{C}^1 \subset \mathbb{C}^2 \subset \ldots$. Given any $X \in G_n(\mathbb{C}^\infty)$ we can define the Schubert symbol $\sigma = (\sigma_1, \ldots, \sigma_n)$ exactly as in the real case, with $\dim(X \cap \mathbb{C}^{\sigma_i - 1}) + 1 = \dim(X \cap \mathbb{C}^{\sigma_i})$. Here we mean complex dimension, of course. The cell $e(\sigma)$ is then defined as the set of all X with Schubert symbol σ. We now show that each of these cells is in fact homeomorphic to a ball of dimension $2\sum_i \sigma_i - i$. To see this, we can re-perform proofs of Lemmas 6.2 and 6.3 on p. 76-79 in this context.

Lemma 6.2$\mathbb{C}$: Each n-plane $X \in e(\sigma)$ has a unique orthonormal basis in $(x_1, \ldots, x_n) \in \prod_{i=1}^n H(\sigma_i)$ where $H(k) = \{v \in \mathbb{C}^k - \mathbb{C}^{k-1} | v_k \in \mathbb{R}^+\}$.

Proof: By the definition of σ_i we see that the set of unit vectors in $X \cap H(\sigma_1)$ is of the form $S = \{\lambda e | e \in H(\sigma_1) \cap X, \lambda \in \mathbb{C}, |\lambda| = |e| = 1\}$. There is a unique element in this set with real first component, i.e $\frac{\bar{e}_{\sigma_1}}{|e_{\sigma_1}|}e$ (not that this is well-defined always because $e_{\sigma_1} \neq 0$). This must be x_{σ_1}. Inductively, we see that the condition that $\langle x_i, x_j \rangle = 0$ for all $i < j$, $x_j \in H(\sigma_j) \cap X$ and $|x_j| = 1$ again restricts x_j to a set of the form S. Then the same argument shows that the condition on the first component uniquely determines x_j. $\square$

Now let $e'(\sigma) = V_n^0(\mathbb{C}^\infty) \cap \prod_{i=1}^n H(\sigma_i)$ and $\bar{e}'(\sigma)$ be its closure $V_n^o(\mathbb{C}^\infty) \cap \prod_{i=1}^n \bar{H}(\sigma_i)$.

Lemma 6.3ℂ: $\bar{e}'(\sigma)$ is topologically a closed cell of dimension $d(\sigma) = 2\sum_i \sigma_i - i$. Furthermore, $q : V_n^o(\mathbb{C}^\infty) \to G_n(\mathbb{C}^\infty)$ maps $e'(\sigma)$ onto $e(\sigma)$ homomorphically.

Proof: As with the proof of 6.3ℝ in the book, we will prove that $\bar{e}'(\sigma)$ is a closed cell of the right dimension with induction on n. If $n = 1$, then $\bar{e}'(\sigma)$ is the set of all vectors $x_1 = (x_{11}, \dots, x_{1\sigma_1}, 0, 0, \dots)$ with $|x|^2 = 1$ and $x_{1\sigma_1} \in \mathbb{R}^+ \cup 0$. This is equivalent to the upper half sphere in $2\sigma_1 - 1$ dimensions, which is $2\sigma_1 - 2$ dimensional. So our base case is covered.

Now the induction. Let b_i be the standard basis of $\mathbb{C}^\infty$, and observe that $b_i \in H^{\sigma_i}$ obviously. Consider the cell $\sigma = (\sigma_1, \dots, \sigma_{n+1})$. By the induction assumption we know that $\sigma' = (\sigma_1, \dots, \sigma_n)$ is a $2\sum_{i=1}^n \sigma_i - i$ dimensional closed cell. Now let D be defined by:

$$D := \{v \in \bar{H}^{\sigma_n} | |v|^2 = 1, v_{\sigma_n} \in \mathbb{R}^+ \cup 0, \langle b_i, v \rangle = 0 \text{ for } 1 \leq i \leq n\}$$

Evidently, by the argument from before and the dimensional restriction of the $\langle b_i, v \rangle$ conditions, D is $2(\sigma_{n+1} - 1) - 2n$ dimensional.

We will now construct a homeomorphism $\bar{e}'(\sigma') \times D \to \bar{e}'(\sigma)$. For this, observe that the map $(u, v, x) \mapsto T(u, v)x$ is still well-defined, with all of the properties that it had before. In particular, the formula on p. 77 still makes sense, Property 1 (continuity) holds and Property 2 holds in the form $T(u, v)x \equiv x \mod \mathbb{C}^k$ if $u, v \in \mathbb{C}^k$. We can thus define $T(x_1, \dots, x_n)$ as the map $T(b_n, x_n) \circ T(b_{n-1}, x_{n-1}) \circ \dots \circ T(b_1, x_1)$. All of the properties of this T are as stated on p. 77 (they're all algebraic orthogonality properties, so they work over $\mathbb{C}$ as well). So we can define our homeomorphism analogously:

$$f((x_1, \dots, x_n), u) = (x_1, \dots, x_n, T(x_1, \dots, x_n)u)$$

And we again have $\langle x_i, Tu \rangle = \langle Tb_i, Tu \rangle = \langle b_i, u \rangle = 0$, $|Tu| = |u| = 1$ and $Tu = u \mod \mathbb{C}^{\sigma_n}$ implying $[Tu]_{\sigma_n} \in \mathbb{R}^+ \cup 0$ and so $f((x_1, \dots, x_n), u) \in \bar{e}'(\sigma)$. The inverse is of course given by $(x_1, \dots, x_{n+1}) \mapsto (x_1, \dots, x_n, T^{-1}(x_1, \dots, x_n)x_{n+1})$ which has formula as on p. 78. As before, a similar induction argument would cover the case where we replace $\bar{e}'(\sigma')$ and $\bar{e}'(\sigma)$ with their unbarred interior counterparts and D with its interior. We simply replace every $\mathbb{R}^+ \cup 0$ with $\mathbb{R}^+$. The same dimension counting and elementary topology arguments used on the bottom of p. 78 and top of p. 79 suffice to show that q is a homeomorphism (our argument for 6.2ℂ showed that it was bijective). $\square$

The result described in this problem now follows exactly from the counting argument used in the real case to finish up Section 6. To calculate the explicit representative of $c_i(\gamma^n)$ as a cocycle, we can replicate the arguments of Problems 6-B, 6-C and 7-A. We won't even repeat the arguments for 6-B and 6-C, because they are exactly the same. The fact that the inclusion map $i^* : H^p(G_n(\mathbb{C}^\infty)) \to H^p(G_n(\mathbb{C}^{n+k}))$ is an isomorphism for $p < k$ follows from a counting argument on the cells, as before. Furthermore, the map $X \mapsto \mathbb{C} \oplus X$ gives an embedding $G_n(\mathbb{C}^m) \to G_{n+1}(\mathbb{C}^{m+1})$ covered by a bundle map $\epsilon^1 \oplus \gamma^n(\mathbb{C}^m) \to$

$\gamma^{n+1}(\mathbb{C}^{m+1})$ and the fact that r-cells map to r-cells corresponding to the same partition is an algebraic fact that doesn't correspond to working over $\mathbb{C}$ or $\mathbb{R}$.

Last, we come to the repetition of 7-A. Here we will literally repeat the argument given in Solution 7-A above. By the isomorphism $H^k(G_n(\mathbb{C}^\infty)) \simeq H^k(G_n(\mathbb{C}^{n+k}))$ via the inclusion cell map, we may reduce to the $w_k(\gamma^n(\mathbb{C}^{n+k}))$. Furthermore, the map $f : \epsilon^1 \oplus \gamma^n(\mathbb{C}^m) \to \gamma^{n+1}(\mathbb{C}^{m+1})$ takes any Schubert cell $e(\sigma)$ corresponding to a *partition* σ to the cell of the same *partition* (note that we are labelling by partitions here). Iterative composition gives us a map $f_j : \epsilon^j \oplus \gamma^n(\mathbb{C}^m) \to \gamma^{n+j}(\mathbb{C}^{m+j})$ which pulls back Chern classes, so $f_j^* c_i(\gamma^{n+j}(\mathbb{C}^{m+j})) = c_i(\gamma^n(\mathbb{C}^m))$.

But now observe that $f_j^* c_k(\gamma^n(\mathbb{C}^{n+k})) = c_k(\gamma^{n-j}(\mathbb{C}^{n+k-j})) = 0$ if $k > n-j$ by dimensionality. But the cells in the image of the map $G_{n-j}(\mathbb{C}^{n+k-j}) \to G_n(\mathbb{C}^{n+k})$ are the cells corresponding to partition of k into $\leq n-j < k$ integers each $\leq k$, so $c_k(\gamma^n(\mathbb{C}^{n+k}))$ must vanish on all cells corresponding to partitions of σ into fewer than k integers. There is only one cell that doesn't satisfy this condition, the cell corresponding to the partition of k into k 1's. So $c_k(\gamma^n)$ is some multiple of the cocycle dual to this cell.

By algebraic independence, we must have $c_k(\gamma^n) = \pm 1(1, \dots, 1)^*$. We reason as so. If it were not, then we would still have $p(c_1, \dots, c_n) = (1, \dots, 1)^*$ (with k 1's) for some polynomial p with integer coefficients. If p contains a non-trivial dependence on c_j for $j \neq k$, this would give a non-trivial relation between the c_i which cannot happen. So this must be of the form $\lambda c_1 = (1, \dots, 1)^*$ But for integer λ that's only possible if $\lambda = \pm 1$. This concludes the proof.

Problem 14-E In analogy with the construction of Chern classes, show that it is possible to define the Stiefel-Whitney classes of a real n-plane bundle inductively by the formula $w_i(\xi) = \pi_0^* w_i(\xi_0)$ for $i < n$. Here the top Stiefel-Whitney class $w_n(\xi)$ must be constructed by the procedure of Section 9, as a mod 2 analogy of the Euler class.

Solution 14-E First we define the $\mathbb{Z}/2$ analogue of the Euler class of an n-plane bundle ξ with total space E, which we will denote as $f(\xi)$, as the image of the Thom class $u \in H^n(E, E_0; \mathbb{Z}/2)$ (which is defined without the orientation in the $\mathbb{Z}/2$ case) via the maps $H^*(E, E_0; \mathbb{Z}/2) \xrightarrow{i^*} H^*(E; \mathbb{Z}/2) \xrightarrow{(\pi^*)^{-1}} H^*(B; \mathbb{Z}/2)$ taking $u \mapsto u|_E \mapsto (\pi^*)^{-1}(u|_E) =: f(\xi)$. Then we can define the Stiefel-Whitney classes recursively as $w_n(\xi) = f(\xi)$ and $w_i(\xi) = (\pi_0^*)^{-1}(w_i(\xi^\perp))$ if $i < n$. Here $\xi^\perp$ is defined as the perpendicular sub-bundle in $\pi_0^*\xi \simeq \epsilon^1 \oplus \xi^\perp$ to the trivial bundle given by pairs $(x, \lambda x)$ in $\pi^*\xi$.

To show that this is well-defined, we use the following fibration long exact sequence for cohomology (with $\mathbb{Z}/2$ coefficients implicit). Assume without loss of generality that B is path connected. Then we have:

$$0 \to H^0(B) \to H^0(E_0) \to \cdots \to H^i(B) \xrightarrow{\pi_0^*} H^i(E_0) \xrightarrow{\mathrm{res}^*} H^i(F_0) \to \cdots \to H^{n-1}(E_0) \to H^{n-1}(F_0) \to H^n(B)$$

This shows that π_0^* is an isomorphism $H^i(B, \mathbb{Z}/2) \to H^i(E_0, \mathbb{Z}/2)$ for $i < n-1$, so the S-W classes are well-defined for this range of i (assuming inductively that they are well-defined for $\xi^\perp$). For $i = n-1$ (the only other case) the exact sequence above shows that $H^{n-1}(B, \mathbb{Z}/2) \to H^{n-1}(E_0, \mathbb{Z}/2)$ is injective. To show that $f(\xi^\perp) = w_{n-1}(\xi^\perp)$ is in the image of π_0^*, it suffices by exactness to show that the restriction morphism $H^{n-1}(E_0) \to H^{n-1}(F_0)$ sends $f(\xi^\perp)$ to 0. But note that the inclusion $i : F_0 \to E_0$ is homotopy equivalent to the inclusion $S^{n-1} \to E_0$ at a fiber, and that this inclusion is covered by a bundle map $\epsilon^n = \epsilon^1 \oplus \tau(S^{n-1}) \to \pi_0^*\xi \simeq \epsilon^1 \oplus \xi^\perp$ which is compatible with the direct sum decomposition. In particular, by the naturally of the image of $f(\xi^\perp)$ under the restriction map is precisely $f(\tau(S^{n-1})$ (under the homotopy equivalence $F_0 \simeq S^{n-1}$).

So it suffices to show that this vanishes. But $H^n(S^{n-1})$ is 1-dimensional so it suffices to show that $\langle f(\tau(S^{n-1})), [M]\rangle \in \mathbb{Z}/2$ is 0. And since τ is orientable, $f(\tau) = e(\tau) \mod 2$ and $\langle f(\tau(S^{n-1})), [M]\rangle = \langle e(\tau), [M]\rangle \mod 2 = \chi(M) \mod 2$. Now simply appreciate that $\chi(S^{n-1})$ is either 0 or 2. So the latter expression vanishes. Thus π_0^* has $w_{n-1}(\xi^\perp)$ in its image and our definition gives a well-defined set of cohomology classes.

Now we prove the axioms of S-W classes using the above formulation.

(1) $w_0(\xi) = 1$ is clear from the definition (since each (π_0^*) is a ring map so 1 goes to 1 in H^0). Also by construction $w_i(\xi) = 0$ for $i > n$ when ξ is an n-plane bundle.

(2) Naturality: If $\bar{f} : X \to Y$ is covered by a bundle map $f : \xi \to \eta$ then $\bar{f}^* w_i(\eta) = w_i(\xi)$. Proof: Induction on n, as in Lemma 14.2. The unoriented Euler class is natural (discussed in Sections 6 and 9) so $f^* w_n(\eta) = w_n(\xi)$. Furthermore, by the induction hypothesis $f_0 : E_0 \to E_0'$ (which is covered by an induced map $f_0 : \xi^\perp \to \eta^\perp$ with the correct choice of metric on both sides, or by the metric-less description of $\xi^\perp$ and $\eta^\perp$) so $f_0^* w_i(\eta^\perp) = w_i(\xi)$. Then since $f_0 \pi_0' = \pi_0 \bar{f}$ (where π_0 and π_0' are the projections for ξ and η respectively) we have that $\bar{f}^* w_i(\eta) = (\pi_0^*)^{-1} f_0^* (\pi_0')^* w_i(\eta) = w_i(\xi)$.

(3) Whitney Product Theorem: This follows from the exact same argument as in the proof of the sum formula for Chern classes on p. 164-167 in the book, and indeed as in the tensor product formula argument in Problem 7-C. Namely, by using the fact that the Whitney sum of two bundles ξ^m and η^n can always be mapped into the bundle $\pi_1^*\gamma^m \oplus \pi_2^*\gamma^n$ over $G_m \times G_n$ and the fact that $H^*(G_m \times G_n) = H^*(G_m) \otimes H^*(G_n) = \mathbb{Z}/2[w_1(\gamma^m) \times 1, \ldots, w_m(\gamma^m) \times 1, w_1(\gamma^n), \ldots, 1 \times w_n(\gamma^n)]$, we know that:

$$w(\pi_1^*\gamma^m \oplus \pi_2^*\gamma^n) = p_{m,n}(w_1(\gamma^m) \times 1, \ldots, w_m(\gamma^m) \times 1, w_1(\gamma^n), \ldots, 1 \times w_n(\gamma^n))$$

To prove the sum formula it thus suffices to prove it for the bundles $\pi_1^*\gamma^m$ and $\pi_2^*\gamma^n$. This can be done inductively (as for the Chern classes), precisely as in p. 166 and 167. The only difference if that you have to prove the triviality of the mod 2 Euler class for a bundle $\xi \oplus \epsilon^1$, for which the proof on p. 97 goes through completely unchanged with $\mathbb{Z}/2$ coefficients.

(4) For the last part, just observe that $\phi^{-1}(u \cup u) = (\pi_0^*)^{-1}(u|_E)$ (that is, without referring to $u \cup u$ as $Sq^n u$, the formulae agree). Thus the proof of $w_1(\gamma_1^1) \neq 0$ on p. 93 is the same. Namely, if E is the total space of γ_1^1 (the Mobius strip), we argue by excision that $H^*(P^2, D^2) \simeq H^*(E, E_0)$ and since $H^1(P^2, D^2) \simeq H^1(P^2)$ via the restriction $H^*(P^2, D^2) \to H^*(P^2)$ we have a map $H^*(E, E_0) \to H^*(P^2)$ sending the fundamental class u to the non-zero generator of $H^*(P^2)$. $a \cup a \neq 0$ by any of our proofs of the cohomology of P^n, so this proves that $\phi^{-1}(u \cup u) = w_1(\gamma_1^1) \neq 0$ (since ϕ is an isomorphism).

Problem 15-A Using Problem 14-B prove that the mod 2 reduction of the Pontrjagin class $p_i(\xi)$ is equal to the square of the Stiefel-Whitney class $w_{2i}(\xi)$.

Solution 15-A This is a quick calculation.

$$p_i(\xi) \mod 2 = c_{2i}(\xi \otimes \mathbb{C}) \mod 2 = w_{4i}(\xi \oplus \xi) = \sum_{k+j=4i} w_j(\xi) \cup w_k(\xi) = w_{2i}(\xi)^2$$

The last equality is because every other cup product in the sum expression appears twice, and thus cancels mod 2.

Problem 15-B Show that $H^*(G_n(\mathbb{R}^\infty))$ is a polynomial ring over Λ generated by the Pontrjagin classes $p_1(\gamma^n), \ldots, p_{\lfloor n/2 \rfloor}(\gamma^n)$. More generally, for any 2-fold covering space $\tilde{X} \to X$ with covering transformation $t : \tilde{X} \to \tilde{X}$, show that $H^*(X, \Lambda)$ can be identified with the fixed point set of the involution t^* of $H^*(\tilde{X}, \Lambda)$.

Solution 15-B We prove a more general statement, for any covering space $\tilde{X} \to X$ with finite deck transformation group G. In such a situation, any map k simplex $\sigma : \Delta^k \to B$ has exactly $|G|$ lifts $\sigma_f : \Delta_k \to \tilde{X}$ to the cover $\pi : \tilde{X} \to X$ of B with the property that $\pi\sigma_g = \sigma$. These lifts are exchanged by the action of $g \in G$ on the fibers. By examining the construction of the boundary map ∂ and the action of G on the simplices, it is clear that $g\partial\sigma = \partial g\sigma$.

By the above reasoning, we thus have a chain map $\pi_* : C_*(\tilde{X}, \Lambda) \to C_*(X, \Lambda)$ given by taking a simplex σ to $\pi \circ \sigma$. This map can also be viewed as the map $\sigma \mapsto [\sigma]$ where $[\sigma]$ is the G orbit of σ (that is, we can view this as a free-module quotient map). The dual map of cottons $\pi^* : C^*(X, \Lambda) \to C^*(\tilde{X}, \Lambda)$ sends a cochain s to the cochain $s \circ \pi$. This map is an injection because if $s(\sigma) \neq t(\sigma)$ then $s(\sigma_f) \neq t(\sigma_f)$ for σ a chain, σ_f any lift and s, t cottons. Any cochain in the image of π^* is clearly invariant under the pullback action of the deck transformations G (that is, $s \circ \pi = s \circ \pi \circ g$ for any $g \in G$) and any G-invariant cochain $t \in C^*(\tilde{X}, \Lambda)$ is the image of the cochain t' defined as $t'(\sigma) = t'(\sigma_f)$ for any chain σ and any lift σ_f of σ. This is well-defined precisely due to the G invariance of t. Thus the image of π^* is exactly the G invariant cottons.

Now, since the G action is a cochain endomorphism (by $g\partial\sigma = \partial g\sigma$ on the chain level, and thus also for the cottons), the map $C^*(X, \Lambda) \to C^*(\tilde{X}, \Lambda)$ descends to the cohomology as a map $H^*(X, \Lambda) \to H^*(\tilde{X}, \Lambda)$. Now we observe the following. First, if $[a] \in H^i(\tilde{X}, \Lambda)$ is a cohomology class with $g[a] = [a]$ for all $g \in G$, then $\frac{1}{|G|}\sum_{g \in G} ga$ is a G-invariant representative of $[a]$. Thus the image of $H^*(X, \Lambda)$ in $H^*(\tilde{X}, \Lambda)$ contains and is contained in the submodule of G invariant elements of $H^*(\tilde{X}, \Lambda)$, so they are equal. Now we just have to prove that $H^*(X, \Lambda) \to H^*(\tilde{X}, \Lambda)$ remains injective. For this, we use the same trick. In particular, suppose that a and b are two G invariant, closed cochains with $[a] = [b]$ in $H^*(\tilde{X}, \Lambda)$. Then we want to show that this implies that $[a] = [b]$ in $H^*(X, \Lambda)$. To show this, it suffices to show that $a - b = \partial c$ for a, b invariant under G and any c implies we can choose c to be invariant. Indeed, this setup implies that $g\partial c = \partial c$, so setting $c' = \frac{1}{|G|}\sum_{g \in G} gc$ we have:

$$a - b = \partial c = \frac{1}{|G|}\sum_{g \in G} g\partial c = \partial\left(\frac{1}{|G|}\sum_{g \in G} gc\right) = \partial c'$$

So the map $H^*(X, \Lambda) \to H^*(\tilde{X}, \Lambda)$ is injective with image equal to the G invariant classes. Thus the first part of this problem is done.

As an application, we observe that under the assumption that Λ contains $1/2$, we have that $H^*(\tilde{G}_n(\mathbb{R}^\infty);\Lambda)$ is freely generated by $p_i(\tilde{\gamma}^n)$ for $i \in \{1,\dots,\lfloor n/2\rfloor\}$ when n is odd and additionally $e(\tilde{\gamma}^n)$ when n is even. The naturality of characteristic classes implies that $\pi^* p_i(\gamma^n) = p_i(\tilde{\gamma}^n)$ and $\pi^* e(\gamma^n) = e(\tilde{\gamma}^n)$ with $\pi : \tilde{G}_n(\mathbb{R}^\infty) \to G_n(\mathbb{R}^\infty)$ the double cover map (which is covered by a map of the tautological bundles). But since this map is injective (by the above argument) and $H^*(\tilde{G}_n(\mathbb{R}^\infty),\Lambda)$ is generated by the Pontrjagin classes, this implies that $H^*(G_n(\mathbb{R}^\infty),\Lambda)$ is also a free polynomial ring over its Pontrjagin classes.

Problem 15-C

Problem 16-A Substituting $-t_i$ for x in the identity $\prod_{i=1}^n (x+t_i) = \sum_{i=0}^n x^i \sigma_{n-i}$ and then summing over i, prove Newton's formula:

$$\sum_{i=0}^n (-1)^{n-i}\sigma_{n-i}s_i = 0$$

(Here I'm adopting the convention that $s_0 = n$ and $\sigma_0 = 1$). This formula can be used inductively to compute the polynomial $s_n(\sigma_1,\dots,\sigma_n)$. Alternatively, taking the logarithm of both sides of the identity $\prod_{i=1}^n (1+t_i) = 1 + \sum_{i=1}^n \sigma_n$, prove Girard's formula:

$$(-1)^k \frac{s_k}{k} = \sum_{\sum_j j i_j = k} (-1)^{\sum_j i_j} \frac{(-1+\sum_j i_j)!}{\prod_j i_j!} \prod_j \sigma_j^{i_j}$$

Solution 16-A The first part is almost immediate. Namely, $s_k(t_1,\dots,t_n) = \sum_i t_i^k$. Thus:

$$\sum_{i=0}^n (-1)^{n-i}\sigma_{n-i}s_i = (-1)^n \sum_{j=1}^n \left(\sum_{i=0}^n (-t_j)^i \sigma_{n-i} \right) = (-1)^n \sum_{j=1}^n \prod_{i=1}^n (t_i - t_j) = 0$$

The last term vanishes because in every summand $\prod_j (t_i - t_j)$ obviously contains $t_j - t_j = 0$. Proving Girard's formula, we see that:

$$\sum_k \frac{(-1)^{k+1}}{k} s_k = \sum_i \sum_k \frac{(-1)^{k+1}}{k} t_i^k = \log(\prod_i (1+t_i)) = \log(1 + \sum_i \sigma_i) = \sum_m \frac{(-1)^{m+1}}{m} (\sum_i \sigma_i)^m$$

Since s_k is a homogeneous degree k polynomial, it must be equal to the sum of the homogeneous order k components in the Taylor expansion on the right. These are precisely the terms in $(\sum_i \sigma_i)^m$ of the form $\prod_{i=1}^n \sigma_i^{j_i}$ with $\sum_i j_i = m$ and $\sum_i i j_i = k$. Now observe that the coefficient of this term in $(\sum_i \sigma_i)^m$ is $\frac{m!}{\prod_i j_i!}$, the number of ways of dividing a set of m objects into n sets of size j_i for $i \in \{1,\dots,n\}$. This is easy to prove combinatorially. Thus we have the following expression:

$$\frac{(-1)^{k+1}}{k} s_k = \sum_{\sum_i i j_i = k} \frac{(-1)^{\sum_i j_i + 1}}{\sum_i j_i} \frac{(\sum_i j_i)!}{\prod_i j_i!} \prod_i \sigma_i^{j_i}$$

And we have verified our formula.

Problem 16-B The *Chern character* $ch(\omega)$ of a complex n-plane bundle ω is defined to be the formal sum:

$$n + \sum_{k=1}^{\infty} s_k(c(\omega))/k! \in H^{\Pi}(B;\mathbb{Q})$$

Show that the Chern character is characterized by additivity:

$$ch(\omega \oplus \omega') = ch(\omega) + ch(\omega')$$

together with the property that $ch(\eta^1)$ is equal to the formal power series $\exp(c_1(\eta^1))$ for any line bundle η^1. Show that the Chern character is also multiplicative:

$$ch(\omega \otimes \omega') = ch(\omega)ch(\omega')$$

Solution 16-B Additivity is proven in Corollary 16.3. That is, since $s_k(c(\omega \oplus \omega')) = s_k(c(\omega)) + s_k(c(\omega'))$, and $\dim(\omega \oplus \omega') = \dim(\omega) + \dim(\omega')$, we have:

$$ch(\omega \oplus \omega') = m + n + \sum_{k=1}^{\infty} s_k(c(\omega \oplus \omega'))/k! = m + n + \sum_{k=1}^{\infty} (s_k(c(\omega)) + s_k(c(\omega')))/k! = ch(\omega) + ch(\omega')$$

Now observe that $s_k(x) = x^k = \sigma_1(x)^k$ over a one variable polynomial ring. So $ch(\eta^1) = 1 + \sum_{k=1}^{\infty} \frac{c(\eta^1)^k}{k!} = \exp(c_1(\eta^1))$.

Finally, we prove the product formula. As in Problem 7-C, by considering the universal model $G_m(\mathbb{C}^\infty) \times G_n(\mathbb{C}^\infty)$ with the universal tensor product bundle $\pi_1^*\gamma^m \otimes \pi_2^*\gamma^n$ and noting that $H^*(G_m(\mathbb{C}^\infty) \times G_n(\mathbb{C}^\infty); \mathbb{Z}) \simeq H^*(G_m(\mathbb{C}^\infty)) \otimes H^*(G_n(\mathbb{C}^\infty); \mathbb{Z})$ canonically[2] we can argue that a universal formula for $s_k(c(\gamma_1^m) \otimes c(\gamma_2^n))$ in terms of $s_i(c(\gamma_1^m))$ and $s_j(c(\gamma_2^n))$ exists and that it is unique (otherwise there would be non-trivial relations between the Chern classes of $G_n(\mathbb{C}^\infty)$). Furthermore, via the isomorphism $H^*((\mathbb{C}P^\infty)^n) \simeq H^*(G_n(\mathbb{C}^\infty))$ we can argue that this universal relation must be equal to whatever unique relation exists for Whitney sums of line bundles (for which $(\gamma^1)^n$ over $(\mathbb{C}P^\infty)^n$ is the universal model).

If $\xi^m = \oplus_i \xi_i^1$ and $\eta^n = \oplus_j \eta_j^1$ then assuming $c_1(\xi_i \otimes \eta_j) = c_1(\xi_i) + c_1(\eta_j)$ (as in the S-W case) we have:

$$ch(\xi^m \otimes \eta^n) = \sum_{i,j} ch(\xi_i \otimes \eta_j) = \sum_{i,j} \exp(c_1(\xi_i \otimes \eta_j)) = \sum_{i,j} \exp(c_1(\xi_i) + c_1(\eta_j))$$

$$= \sum_{i,j} \exp(c_1(\xi_i)) \exp(c_1(\eta_j)) = \sum_{i,j} ch(\xi_i) ch(\eta_j) = ch(\xi^m) ch(\eta^n)$$

This gives a universal relation as mentioned above, so this formula must hold in general. Thus we merely have to prove the formula $c_1(\xi_i \otimes \eta_j) = c_1(\xi_i) + c_1(\eta_j)$. We already know that $c_1(\xi \otimes \eta) = p_{1,1}(\xi, \eta)$. $p_{1,1}(x, y)$ must be linear and symmetric, so it must be of the form $c(x + y)$. Furthermore, when $y = 0$ we must have $p_{1,1}(x, 0) = x$ because $c_1(\xi \otimes \epsilon^1) = p_{1,1}(c_1(\xi), 0) = c_1(\xi)$. Thus $p_{1,1}(x, y) = x + y$.

[2]There is no torsion here because the complex projective space cohomology vanishes in odd dimensions.

Problem 16-C If $2i_1, \ldots, 2i_r$ is a partition of $2k$ into even integers, show that the $4k$-dimensional characteristic class $s_{2i_1,\ldots,2i_r}(c(\omega))$ of a complex vector bundle is equal to the characteristic class $s_{i_1,\ldots,i_2}(p(\omega_{\mathbb{R}}))$ of its underlying real vector bundle. As examples, show that the $4k$-dimensional class $s_{2,\ldots,2}(c(\omega))$ is equal to $p_k(\omega_{\mathbb{R}})$ and show that the characteristic number $s_{2n}(c)[K^{2n}]$ of a complex manifold is equal to $s_n(p)[K^{2n}]$.

Solution 16-C For partition $I = i_1, \ldots, i_r$ abbreviate $2i_1, \ldots, 2i_r$ as $2I$, for the tuple $\vec{t} = (t_1, \ldots, t_r)$ abbreviate $(t_1^2, \ldots, t_r^2)$ as $\vec{t}^2$ and abbreviate the tuple $(\sigma_1(t_1, \ldots, t_n), \ldots, \sigma_n(t_1, \ldots, t_n))$ as $\sigma(\vec{t})$. Observe that s_I and s_{2I} correspond to the polynomials $\sum_{\sigma\in\Sigma_r}\prod_j t_{\sigma(j)}^{i_j}$ and $\sum_{\sigma\in\Sigma_r}\prod_j t_{\sigma(j)}^{2i_j}$ respectively. In other words, $s_{2I}(\vec{t}) = s_I(\vec{t}^2)$ and thus $s_{2I}(\sigma(\vec{t})) = s_I(\sigma(\vec{t}^2))$. Now because $\sigma_i(\vec{t}^2)$ is a symmetric polynomial, it admits a polynomial expression $\sigma_i(\vec{t}^2) = q_i(\sigma(\vec{t}))$ in terms of $\sigma(\vec{t})$. Thus $s_{2I}(\sigma) = s_I(q(\sigma))$

Now, if we prove that $p_i(\omega_{\mathbb{R}}) = q_i(c(\omega))$ then by the above reasoning, $s_{2I}(c(\omega)) = s_I(q(c(\omega))) = s_I(p(\omega_{\mathbb{R}}))$. To see this, observe that by Corollary 15.5, we have $\sum_i(-1)^i p_i(\omega_{\mathbb{R}}) = (\sum_i(-1)^i c_i(\omega))(\sum_i c_i(\omega))$. Replacing $c_i(\omega)$ with σ_i, we see that this expression reads:

$$\sum_i(-1)^i p_i(\omega_{\mathbb{R}}) = (\sum_i(-1)^i\sigma_i)(\sum_i \sigma_i)) = \prod_j(1-t_j)\prod_j(1+t_j)$$

$$= \prod_j(1-t_j^2) = \sum_i(-1)^i\sigma_i(\vec{t}^2) = \sum_i(-1)^i q_i(c(\omega))$$

By counting degrees of the homogeneous summands, this produces the desired result. The applications are trivial. For the first one, by the above, $s_{2,\ldots,2}(c(\omega)) = s_{1,\ldots,1}(p(\omega_{\mathbb{R}})$ and since $s_{1,\ldots,1}$ is by definition $\sum_{\sigma\in\Sigma_n}\prod_{j=1}^k t_{\sigma(j)}$, which is evidently the kth symmetric polynomial, we have $s_{2,\ldots,2}(c(\omega)) = s_{1,\ldots,1}(p(\omega_{\mathbb{R}}) = p_k(\omega_{\mathbb{R}})$. In the second application, we have $s_{2n}(c(\omega)) = s_n(p(\omega))$ so $\langle s_{2n}(c(\omega)), [M]\rangle = \langle s_n(c(\omega)), [M]\rangle$ and these are equal to the characteristic numbers by definition.

Problem 16-D If the complex manifold K^n is complex analytically embedded in K^{n+1} with dual cohomology class $u \in H^2(K^{n+1}, \mathbb{Z})$ show that the total tangential Chern class $c(K^n)$ is equal to the restriction to K^n of $c(K^{n+1})/(1+u)$. For any cohomology class $x \in H^{2n}(K^{n+1}, \mathbb{Z})$ show that the Kronecker index $\langle x|_{K^n}, \mu_{2n}\rangle$ is equal to $\langle xu, \mu_{2n+2}\rangle$. Using these constructions, compute $c(K^n)$ for a non-singular algebraic hyper surface of degree d in$\mathbb{C}P^{n+1}$ and prove that the characteristic number $s_n[K^n]$ is equal to $d(n+2-d^n)$.

Solution 16-D First observe that by Theorem 11.3 we have $u|_{K^n} = e(\nu)$ with $e(\nu)$ the Euler class of the normal bundle of K^n. Since we have $\tau^{n+1} = \tau^n \oplus \nu^1$ for τ^n the tangent bundle of K^n, τ^{n+1} the tangent bundle of K^{n+1} restricted to τ^{n+1} and ν^1 the normal bundle, we can write:

$$c(\tau^{n+1})|_{K^n} = c(\tau^{n+1}|_{K^n}) = c(\tau^n \oplus \nu^1) = c(\tau^n)c(\nu^1) = c(\tau)(1+e(\nu^1)) = c(\tau)((1+u)|_{K^n})$$

Dividing both sides by $(1+u)|_{K^n}$ we get the first desired formula. The second formula is an application of Problem 11-C. In particular, we have:

$$\langle xu, \mu_{K^{n+1}}\rangle = \langle x, u \cap \mu_{K^{n+1}}\rangle = \langle x, i_*\mu_{K^n}\rangle = \langle x|_{K^n}, \mu_{K^n}\rangle$$

For an embedded hyper surface K^n in $\mathbb{C}P^{n+1}$ that is the 0-set of a homogeneous polynomial of degree d, we have $u = da$, where $a = c_1((\gamma^1)^v)$, i.e the first Chern class of $O(1)$ in the Picard group. To see this, we observe that a homogeneous polynomial of degree d can be viewed as a section of the bundle $((\gamma^1)^v)^{\otimes d}$. The normal bundle is isomorphic to the restriction of this bundle to K^n, i.e the pullback of $((\gamma^1)^v)^{\otimes d}$ via $i : K^n \to \mathbb{C}P^{n+1}$. Thus by our tensor product and dual formulae for the Chern class, we see that $c_1(\nu) = i_*(-dc_1(\gamma^1)) = da|_{K^n}$. The more formal way to get this result is through adjunction. Now observe that by the formulae above we have:

$$c(K^n) = \frac{(1+a)^{n+2}}{1+da}|_{K^n}$$

Furthermore, if we recall that $s_n(c(\omega))$ could be expressed in terms of the nth order term in the formal power series of $\log(c(\omega))$, i.e log of the total Chern class, then we may observe the following identity (which holds on the level of formal power series):

$$\log(c(K^n)) = \log(\frac{(1+a)^{n+2}}{1+da}) = (n+2)\log(1+a) - \log(1+da)$$

But examining the nth coefficient of the latter expression we see that $\frac{(-1)^n}{n} s_n(c(K^n)) = \frac{(-1)^n}{n}((n+2) - d^n)(a|_{K^n})^n$. Thus we have:

$$\langle s_n(c(K^n)), \mu_{K^n}\rangle = d(n+2-d^n)\langle a^{n+1}, \mu_{\mathbb{C}P^{n+1}}\rangle = d(n+2-d^n)$$

Problem 16-E Similarly, if $H_{m,n}$ is a non-singular hyper-surface of degree $(1,1)$ in the product $\mathbb{C}P^m \times \mathbb{C}P^n$ of complex projective spaces with $m, n \geq 2$, prove that the characteristic number $s_{m+n-1}[H_{m,n}]$ is equal to $-(m+n)!/m!n!$. Using disjoint union of hyper-surfaces, prove that for each dimension n there exists a complex manifold K^n with $s_n[K^n] = p$ if $n+1$ is a power of the prime p, or with $s_n[K^n] = 1$ if $n+1$ is not a prime power.

Solution 16-E Using the fact that $H^*(\mathbb{C}P^m \times \mathbb{C}P^n; \mathbb{Z}) \simeq H^*(\mathbb{C}P^m; \mathbb{Z}) \otimes H^*(\mathbb{C}P^n; \mathbb{Z})$ (in particular due to the fact that the even cohomology groups vanish, thus so does the torsion in this tensor product) we have:

$$H^2(\mathbb{C}P^m \times \mathbb{C}P^n; \mathbb{Z}) \simeq H^2(\mathbb{C}P^m) \oplus H^2(\mathbb{C}P^n)$$

Thus $H^2(\mathbb{C}P^m \times \mathbb{C}P^n; \mathbb{Z})$ is the rank 2 $\mathbb{Z}$ module generated by $a_m \times 1$ and $1 \times a_n$ (with a_m and a_n defined analogously to a as in 16-D). The degree $(1,1)$ assumption implies that the fundamental class is $a_m \times 1 + 1 \times a_n$. This, in addition to the identity $c(\tau(\mathbb{C}P^m \times \mathbb{C}P^n)) = c(\pi^*\tau(\mathbb{C}P^m))c(\pi^*\tau(\mathbb{C}P^n)) = (1 + a_m \times 1)^m(1 + 1 \times a_n)^n$ yields:

$$c(H_{m,n}) = \frac{(1 + a_m \times 1)^{m+1}(1 + 1 \times a_n)^{n+1}}{1 + a_m \times 1 + 1 \times a_n}|_{H_{m,n}}$$

Now we calculate s_{m+n-1} as the $m+n-1$-th term in the expansion of $\log(c(H_{m,n}))$:

$$\log(c(H_{m,n})) = (m+1)\log(1 + a_m \times 1) + (n+1)\log(1 + 1 \times a_n) - \log(1 + a_m \times 1 + 1 \times a_n)$$

The order $m+n-1$ part of this expansion is (after normalization):

$$s_{m+n-1}(c(H_{m,n})) = (m(a_m^{m+n-1} \times 1) + (n+1)(1 \times a_n^{m+n-1}) - (a_m \times 1 + 1 \times a_n)^{m+n-1})|_{H_{m,n}} =$$

But observe that by dimensionality $a_m^{m+n-1} = a_n^{m+n-1} = 0$ so we just have $s_{m+n-1}(c(H_{m,n})) = -(a_m \times 1 + 1 \times a_n)^{m+n-1})|_{H_{m,n}}$. Then by the results of 16-D we have:

$$s_{m+n}[H_{m,n}] = \langle s_{m+n-1}(c(H_{m,n}))u, \mu_{\mathbb{C}P^m \times \mathbb{C}P^n} \rangle = \langle -(a_m \times 1 + 1 \times a_n)^{m+n}, \mu_{\mathbb{C}P^m} \times \mu_{\mathbb{C}P^n} \rangle$$

The terms in the binomial expansion of the expression $-(a_m \times 1 + 1 \times a_n)^{m+n}$ are of the form $-\binom{m+n}{i} a_m^i \times a_n^j$ for $i+j = m+n$. The only non-zero term of this form is $-\frac{(m+n)!}{m!n!} a_m^m \times a_n^n$. Then we have:

$$s_{m+n}[H_{m,n}] = -\frac{(m+n)!}{m!n!} \langle a_m^m \times a_n^n, \mu_{\mathbb{C}P^m} \times \mu_{\mathbb{C}P^n} \rangle = -\frac{(m+n)!}{m!n!} \langle a_m^m, \mu_{\mathbb{C}P^m} \rangle \langle a_n^n, \mu_{\mathbb{C}P^n} \rangle = -\frac{(m+n)!}{m!n!}$$

Now observe that if we consider a degree $(1,1)$ hyper-surface K^n in $\mathbb{C}P^n \times \mathbb{C}P^1$ with $n+1 = p^k$ a prime power, we have $s_n[K^n] = -\frac{p^k!}{(p^k-1)!} = -p^k$. Furthermore, if we take $m = p^{k-1}$, and let L^n be a degree $(1,1)$ hyper-surface in $\mathbb{C}P^m \times \mathbb{C}P^m$, then $s_n[L^m] = -\frac{p^k!}{p^{k-1}!(p^k - p^{k-1})!}$. The latter expression contains only one factor of p in its prime factorization. Indeed, we observe that it is equal to $\frac{p^k \cdot p^k - 1 \cdot \ldots (p^k - p^{k-1} + 1)}{p^{k-1} \cdot \ldots \cdot 1}$. Furthermore, $p^k - p^{k-1} + j$ has the same number of factors of p in its prime factorization as j, if $0 < j < p^{k-1}$. In this case $j = qp^l$ for some q coprime to p and $l < k-1$, so $p^k - p^{k-1} + j = p^l(p^{k-l} - p^{k-l-1} + q)$ where the $p^{k-l} - p^{k-l-1} + q = q \neq 0 \mod p$. Thus in $\frac{p^k \cdot p^k - 1 \cdot \ldots (p^k - p^{k-1} + 1)}{p^{k-1} \cdot \ldots \cdot 1}$ all of the p factors cancel except for the factors coming from p^k / p^{k-1}, of which there is evidently one.

This all implies that the gcd of $s_n[L^m]$ and $s_n[K^n]$ is p. Thus by a Euclidean algorithm, there exists a and b such that $as_n[L^m] + bs_n[K^n] = p$. Setting $M^n = aK^n + bL^n$ where $+$ is interpreted as disjoint union and a negative a or b implies orientation reversal, we get our result.

If $n+1$ is not a power of p, then we get a similar result as so. Let $n+1 = \prod_i p_i^{k_i}$ and let $m_i = p_i^{k_i}$. Then by the same argument as above, $\binom{n+1}{p_i^{k_i}}$ is coprime to p_i. Indeed, in this case we have $\binom{n+1}{p_i^{k_i}} = \frac{n+1 \cdot n \cdot \ldots (n - p_i^{k_i} + 1)}{p_i^{k_i} \cdot \ldots \cdot 1}$, and $n - p_i^{k_i} + j$ having the same number of p_i factors as j when $0 < j < p_i^{k_i}$. This time, however, the p_i factors in the last part cancel as well, leaving the result p_i free. Let K_i^n be a $(1,1)$ hyper-surface embedded in $\mathbb{C}P^{m_i} \times \mathbb{C}P^{n-m_i+1}$ and let K_0^n be a $(1,1)$ hyper-surface embedded in $\mathbb{C}P^n \times \mathbb{C}P^1$. Then by these arguments and the fact that $\binom{n+1}{n} = n+1$, we have that $s_n[K_0^n] = n+1$ and $s_n[K_i^n]$ is coprime to p_i. Thus by an iterated Euclidean algorithm we can find a_i, $i = 0, \ldots, n$, with $s_n[\sum_i a_i K_i^n] = \sum_i a_i s_n[K_i^n] = 1$. This produces the last part of this problem.

Problem 16-F

Problem 18-A As in the proof of 18.5, suppose that f has the origin as a regular value throughout some compact $K'' \subset W \subset \mathbb{R}^m$. If g is uniformly close to f and the derivatives $\partial g_i / \partial x_j$ are uniformly close to $\partial f_i / \partial x_j$, show that g has the origin as a regular value throughout K''.

Solution 18-A First observe that the condition of being full rank is an open condition on $m \times n$ matrices. Thus at each point p with Df_p full rank, there exists an $\epsilon_p > 0$ such that $B_{\epsilon_p}(Df_p)$ contains only maximal rank matrices. Relatedly, Df is uniformly continuous on K' so we can pick a uniform $C > 0$ with $|p-q| < \epsilon$ implies $|Df_p - Df_q| < C\epsilon$.

Now apply this to our situation. Consider the compact set $S = K' \cap f^{-1}(0)$. By assumption all $p \in S$ have Df_p maximal rank. Thus let $\epsilon := \min_{p \in S} \epsilon_p$. If we define $U_k = f^{-1}(\mathbb{R}^n - \bar{B}_{1/k}(0))$ observe that $\{U_k \cap K'\} \cap \{B_{\epsilon/4C}(f^{-1}(0)) \cap K'\}$ is an open cover of K' which is compact. In particular, $f^{-1}(B_{1/k}(0)) \subset B_{\epsilon/4C}(f^{-1}(0))$ for some large k.

Now pick $\delta = \min(1/k, \epsilon/4)$ with k as above and suppose that $\|f - g\|_{C^1} < \delta$. Consider a $q \in g^{-1}(q) \cap K'$. Since $|f(q) - g(q)| < 1/k$ we know that $|f(q)| < 1/k$. Thus by our choice of k, there exists a $p \in f^{-1}(0) \cap K'$ with $|q - p| < \epsilon/4C$. Then we see that for this p we have:

$$|Dg_q - Df_p| < |Dg_q - Df_q| + |Df_q - Df_p| < \delta + C|p - q| < \delta + \epsilon/4 < \epsilon$$

Then by our definition of ϵ this implies that Dg_q must be full rank. This proves the statement.

Problem 19-A Let $\{T_n\}$ be the multiplicative sequence of polynomials belonging to the power series $f(t) = t/(1 - e^{-t})$. Then the *Todd genus* of a complex n-dimensional manifold is defined to be the characteristic number $\langle T_n(c_1, \dots, c_n), \mu_{2n} \rangle$. Prove that $T[\mathbb{C}P^n] = +1$ and prove that $\{T_n\}$ is the only multiplicative sequence with this property.

Solution 19-A We recall that $K^n = \mathbb{C}P^n$ has $c(K^n) = (1+a)^{n+1}$ for a a certain generator of $H^1(K^n; \mathbb{Z})$. Thus for the Todd class we have $T[K^n] = \frac{a^{n+1}}{(1-e^{-a})^{n+1}}$. Here T_n will be the nth order term in cohomology, i.e the coefficient of a^n in the Tayloe series at 0. Thus $T[K^n] = \langle T_n[K^n]a^n, \mu_{K^n} \rangle = T_k[K^n]$ because $\langle a^n, \mu_{2n} \rangle = 0$. To get this term we can do a contour integral about 0 as so:

$$T[K^n] = \frac{1}{2\pi i} \int \frac{a^{n+1}}{a^{n+1}(1 - e^{-a})^{n+1}} da = \frac{1}{2\pi i} \int \frac{1 + u + u^2 + \dots}{u^{n+1}} du = 1$$

Here we use the change of variables $u = 1 - e^{-a}$ which has $du = e^{-a} da = (1 - u) da$ thus $da = \frac{1}{1-u} du = (1 + u + u^2 + \dots) du$. Now observe that this procedure will generally get the order a^n coefficient $f_n^{(n)}$ of $f(a)^n$ for any Taylor series $f(a)$. But we see that $f(a)^n$ has Taylor expansion:

$$f(a)^n = (\sum_{i=0}^{n} \lambda_i a^i)^n = \sum_{i=0}^{\infty} (\sum_{\sum i_j = i} \prod_j \lambda_{i_j}) a^i$$

In particular, since $\lambda_0 = 1$ we have $\lambda_n = f_n^{(n)} - p_n(\lambda_1, \dots, \lambda_n)$ for some polynomials p_n. Thus inductively, we see that $f_n^{(n)} = 1$ for all n fixes λ_n and thus the Taylor expansion of $f(a)$. Therefore $f(a)$ is determined by the property that $T[K^n] = 1$ for all n, and thus its corresponding multiplicative sequence is determined as well.

Problem 19-B If $\{K_n\}$ is the multiplicative sequence belonging to $f(t) = \sum_i \lambda_i t^i$, let us indicate the dependence on the coefficients λ_i by setting $K_n(x_1, \ldots, x_n) = k_n(\lambda_1, \ldots, \lambda_n, x_1, \ldots, x_n)$ where k_n is a polynomial with integer coefficients. By considering the case where $\lambda_1, \ldots, \lambda_n$ are elementary symmetric polynomials in n indeterminates, prove the symmetry property $k_n(x_1, \ldots, x_n, \lambda_1, \ldots, \lambda_n)$. In particular, prove that the coefficients of $x_{i_1}, \ldots, x_{i_r}$ in the polynomial $K_n(x_1, \ldots, x_n)$ is equal to $s_{i_1,\ldots,i_r}(\lambda_1, \ldots, \lambda_n)$.

Solution 19-B Observe the following. Given $f(a) = \sum_i \lambda_i a^i$ define a multiplicative sequence K' by $K'_n(a) = \sum_I a_I s_I(\lambda_1, \ldots, \lambda_n)$ where I is a partition of n. If we show that this multiplicative sequence has $K'(1+a) = f(a)$ and is i fact multiplicative, the we will have established that$K' = K$, where K is the usual multiplicative sequence corresponding to f, in which case it will follow that the coefficient of a_I in the usual K is $s_I(\lambda_1, \ldots, \lambda_n)$.

First, we see that $K'(1+a) = f(a)$ obviously. Indeed, the only non-zero a_I is a_1^n, which corresponds to $s_{1,\ldots,1}(\lambda_1, \ldots, \lambda_n) = \lambda_n$. So by our definition $K'(1+a) = \sum_i a^i s_{1,\ldots,1}(\lambda_1, \ldots, \lambda_n) = \sum_i \lambda_i a^i = f(a)$. The second part is also obvious once written down. Observe:

$$K(ab) = \sum_I (ab)_I s_I = \sum_I (ab)_I \sum_{GH=I} s_G s_H = \sum_I \sum_{GH=I} a_G b_H s_G s_H$$

Problem 19-C Using Cauchy's identity:

$$f(t)\frac{d}{dt}(\frac{t}{f(t)}) = 1 - t\frac{d\log(f(t))}{dt} = 1 + \sum (-1)^j s_j(\lambda_1, \ldots, \lambda_j)t^j$$

prove that the coefficient of p_n in the L-polynomial $L_n(p_1, \ldots, p_n)$ is equal to $2^{2k}(2^{2k-1}-1)B_k/(2k)! \neq 0$.

Solution 19-C Observe that for the multiplicative sequence L_n we have $f(t) = \frac{\sqrt{t}}{\tanh(t)}$. Thus Cauchy's identity reads:

$$1 + \sum_j (-1)^j s_j(\lambda_1, \ldots, \lambda_j)t^j = \frac{\sqrt{t}}{\tanh(\sqrt{t})}\frac{d}{dt}(\sqrt{t}\tanh(t)) = \frac{1}{2} + \frac{1}{2} \cdot \frac{2\sqrt{t}}{\sinh(2\sqrt{t})}$$

Now applying the formula for the Taylor expansion of $\frac{t}{\sinh(t)}$ discussed on p. 282 (which is trivial to derive using the trigonometric identity $\frac{1}{\sinh(2t)} = \frac{1}{\tanh(t)} - \frac{1}{\tanh(2t)}$ in tandem with the Taylor expansion of $\frac{t}{\tanh(t)}$ from p. 281) we find that:

$$1 + \sum_j (-1)^j s_j(\lambda_1, \ldots, \lambda_j)t^j = 1 + \sum_{i=1}^{\infty} (-1)^i 2^{2i}(2^{2i-1}-1)\frac{B_i}{(2i)!}t^i$$

Matching terms, we find that $s_j(\lambda_1, \ldots, \lambda_n) = 2^{2j}(2^{2j-1}-1)\frac{B_j}{(2j)!}$. Furthermore, by the formula derived in 19-B saying that the x_I coefficient in K_n is $s_I(\lambda)$ implies that $s_n(\lambda)$ is the coefficient of x_n in K_n. So we have our result.

Problem 20-A Let τ be the tangent bundle of the quaternion projective space $\mathbb{H}P^m$. Using the isomorphism $\tau \simeq \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma^{\perp})$ of real vector bundles show that $\tau \oplus \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma) \simeq \mathrm{Hom}_{\mathbb{H}}(\gamma, \mathbb{H}^{m+1})$ and hence that $p(\tau) = (1+u)^{2m+2}/(1+4u)$.

Solution 20-A We reproduce the reasoning of the real (S-W) and complex (Chern) cases. Since $\mathrm{Hom}_{\mathbb{H}}(\cdot, \cdot)$ distributes over direct sum bilinearly for $\mathbb{H}$ vector spaces via a natural isomorphism on the fiber level, we have the following string of isomorphisms:

$$\tau \oplus \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma) \simeq \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma^{\perp}) \oplus \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma) \simeq \mathrm{Hom}_{\mathbb{H}}(\gamma, \gamma^{\perp} \oplus \gamma) \simeq \mathrm{Hom}_{\mathbb{H}}(\gamma, \mathbb{H}^{m+1}) \simeq \oplus_1^m \mathrm{Hom}_{\mathbb{H}}(\gamma, \mathbb{H})$$