Math 55 Section 101 Quiz 5

Problem 1 (4 pt) Let T be the set of polynomials with even integer coefficients and even powers of x. Examples of such polynomials are $2x^4 + 6x^2$ or $16x^{32} - 2x^2 + 2$. $2x^3 + 4x^2$ is not allowed because x^3 is an odd power of x. $3x^2 + 2$ is not allowed because x^2 has an odd coefficient. **Give a recursive definition for T. Solution:** There are many possible definitions. Here is one.

Base Object: $2 \in T$.

Recursive Property: If $p, q \in T$ then $p + q \in T, p - q \in T$ and $x^2 p, x^2 q \in T$.

Problem 2 The Fibonacci sequence is defined recursively as $f_0 = 0, f_1 = 1$ and $f_{i+2} = f_{i+1} + f_i$. Using strong induction, prove the following formula:

$$f_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$$

2.A (2 pt) What is "$P(n)$" in this case?

Solution: I would use $P(n)$ defined as:

$$f_n = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$$

2.B (2 pt) Prove the base case. (Caution: There should be 2 "base cases" in this situation.)

Solution: Here you need to prove $P(0)$ and $P(1)$. We see that:

$$\frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^0 - (\frac{1-\sqrt{5}}{2})^0\right) = \frac{1}{\sqrt{5}}(1-1) = 0 = f_0$$

$$\frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^1 - (\frac{1-\sqrt{5}}{2})^1\right) = \frac{1}{\sqrt{5}}(\sqrt{5}) = 1 = f_1$$

2.C (2 pt) Do the induction step (on the back please). Hint: Notice that $(\frac{1+\sqrt{5}}{2})^2 = 1 + \frac{1+\sqrt{5}}{2}$. **Solution:** Assume $P(k)$ for $k < n$. Then by the definition of the Fibonacci sequence and our induction hypothesis we have:

$$f_n = f_{n-1} + f_{n-2} = \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^{n-1} - (\frac{1-\sqrt{5}}{2})^{n-1}\right) + \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^{n-2} - (\frac{1-\sqrt{5}}{2})^{n-2}\right)$$

$$\frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-2}(\frac{1+\sqrt{5}}{2} + 1) - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n-2}(\frac{1-\sqrt{5}}{2} + 1)$$

Now observe that $(\frac{1+\sqrt{5}}{2})^2 = \frac{6+2\sqrt{5}}{2} = 1 + \frac{1+\sqrt{5}}{2}$ and similarly $(\frac{1-\sqrt{5}}{2})^2 = \frac{6-2\sqrt{5}}{2} = 1 + \frac{1-\sqrt{5}}{2}$. Thus:

$$f_n = \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-2}(\frac{1+\sqrt{5}}{2}+1) - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n-2}(\frac{1-\sqrt{5}}{2}+1)$$

$$= \frac{1}{\sqrt{5}}(\frac{1+\sqrt{5}}{2})^{n-2}(\frac{1+\sqrt{5}}{2})^2 - \frac{1}{\sqrt{5}}(\frac{1-\sqrt{5}}{2})^{n-2}(\frac{1-\sqrt{5}}{2})^2$$

$$= \frac{1}{\sqrt{5}}\left((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n\right)$$

Thus we have verified the induction step.