Introduction to Legendrian knots and Legendrian contact homology

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Abstract

This notes are an introduction to contact homology in its simplest form: the Chekanov's (combinatorial) differential graded algebra associated to a Legendrian knot in $\mathbb{R}^3$. After a brief introduction to the basics of contact topology we focus on Legendrian knots in $\mathbb{R}^3$ and introduce their classical invariants and Chekanov's dga. Finally we will compare Chekanov's combinatorial construction with its geometric motivation coming from holomorphic curves. There is no claim of originality on the material covered in these notes: when no reference is given the only reason is the laziness of the author in tracking back the original source.

1 Introduction

This manuscript is an expanded version of my lecture notes for the course on Legendrian knots and Chekanov's contact homology that I gave in Paris[1] and Meknes[2]. These notes are heavily based on Etnyre's survey on the same subject [6] with some inputs from Geiges's book [13], Chekanov's original paper [2], and Etnyre, Ng and Sabloff's paper [10]. I gratefully thank Baptiste Chantraine, Joan Licata and Vera Vértesi for their help in the preparation of these notes.

[1]Summer school "Homologies d'entrelacs", from 29 June to 3 July 2009

[2]CIMPA school "Low dimensional topology and symplectic geometry", from 21 May to 1 June 2012

2 Classical theory of Legendrian knots

2.1 Basic contact topology

Informally speaking, a contact structure on a $(2n+1)$-manifold is a maximally non-integrable hyperplane distribution. More formally:

Definition 2.1. A 1-form on a $(2n+1)$-dimensional manifold M is called a *contact form* if $\alpha \wedge (d\alpha)^n$ is a volume form on M; that is

$$\forall p \in M, \quad \alpha_p \wedge (d\alpha_p)^n \neq 0. \tag{1}$$

An equivalent way to express the contact condition (1) is to say that α is a contact form if

$$\forall p \in M, \quad (d\alpha_p|_{\ker \alpha_p})^n \neq 0, \tag{2}$$

i.e. $d\alpha|_{\ker\alpha}$ is a symplectic form on $\ker\alpha$.

Definition 2.2. A hyperplane distribution ξ is a *contact structure*[3] on M if it is the kernel of a contact form. We say that (M, ξ) is a contact manifold.

A contact structure can be defined by many contact forms, and any two of them differ by multiplication by a nowhere vanishing function. If $\alpha' = f\alpha$ is another contact form for the same contact structure, then

$$\alpha' \wedge d\alpha' = f^{n+1}(\alpha \wedge d\alpha).$$

This implies that the contact condition does not depend on the choice of the contact form. Moreover M must be orientable and ξ determines a preferred orientation when n is odd.

The contact condition $\alpha\wedge(d\alpha)^n \neq 0$ is a *non-integrability* condition:

Lemma 2.3. *If $N \subset M$ is an m-dimensional submanifold such that $TN \subset \xi|_N$, then $m \leq n$.*

Proof. If $i\colon N \to M$ is the inclusion, then $i^*(d\alpha) = d(i^*\alpha) = 0$ so $d\alpha|_{\ker\alpha} = 0$. Therefore T_pN is an isotropic subspace[4] of ξ_p for all $p \in N$, then the contact condition (2) implies that $\dim N \leq \frac{1}{2}\dim\xi = n$. □

Definition 2.4. Let (M, ξ) an $(n+1)$-dimensional contact manifold. A submanifold $N \subset M$ is called *Lagrangian* if $T_pN \subset \xi_p$ for all $p \in N$ and $\dim N = n$.

[3] Often also called a *cooriented contact structure*

[4] An isotropic subspace of a symplectic vecotr space is a subspace that is contained in its symplectic orthogonal. By simple linear algebra, isotropic subspaces have dimension at most half of the dimension of the total space.

If α is a contact form, there is a unique vector field R solving the system:

$$\begin{cases} \alpha(R) = 1 \\ \iota_R d\alpha = 0 \end{cases} \tag{3}$$

Definition 2.5. The vector field R solving Equation (3) is called the *Reeb vector field* of α.

Lemma 2.6. *Let φ_t be the flow generated by R. Then $\varphi_t^*\alpha = \alpha$ for all $t \in \mathbb{R}$.*

Proof. By Cartan's formula we have:

$$\frac{d}{dt}\varphi_t^*\alpha = \varphi_t^*(\mathcal{L}_R\alpha) = \varphi_t^*(d\iota_R\alpha + \iota_R d\alpha) = 0.$$

Since φ_0 is the identity, the lemma follows. □

Example 2.7. We take coordinates $(x_1, \dots, x_n, y_1, \dots, y_n, z)$ on $\mathbb{R}^{2n+1}$. The standard contact structure on $\mathbb{R}^3$ is the contact structure ξ_{st} defined by the contact form

$$\alpha_{st} = dz - \sum_{i=1}^{n} y_i dx_i.$$

Example 2.8. There is another natural contact form α'_{st} on $\mathbb{R}^{2n+1}$:

$$\alpha'_{st} = dz + \sum_{i+1}^{n} x_i dy_i - y_i dx_i.$$

Exercise 2.9. Let us introduce the notation $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$. Check that the diffeomorphism $f\colon \mathbb{R}^{2n+1} \to \mathbb{R}^{2n+1}$

$$f(\mathbf{x}, \mathbf{y}, z) = \left(\frac{\mathbf{x}+\mathbf{y}}{2}, \frac{\mathbf{y}-\mathbf{x}}{2}, z + \frac{\mathbf{x}\cdot\mathbf{y}}{2}\right)$$

satisfies $f^*\alpha'_{st} = \alpha_{st}$.

Example 2.10. Regard S^{2n+1} as the unit sphere in $\mathbb{R}^{2n+2}$ with coordinates $(x_1, y_1, \dots, x_n, y_n)$. The standard contact structure in S^{2n+1} is the contact structure ξ_{st} defined by the restriction to TS^{2n+1} of the 1-form $\sum_{i=1}^{n+1} x_i dy_i - y_i dx_i$.

Exercise 2.11. Verify that this is a contact form, and that its Reeb vector field generates an S^1 action on S^{2n-1} whose orbits are the fibres of the Hopf fibration.

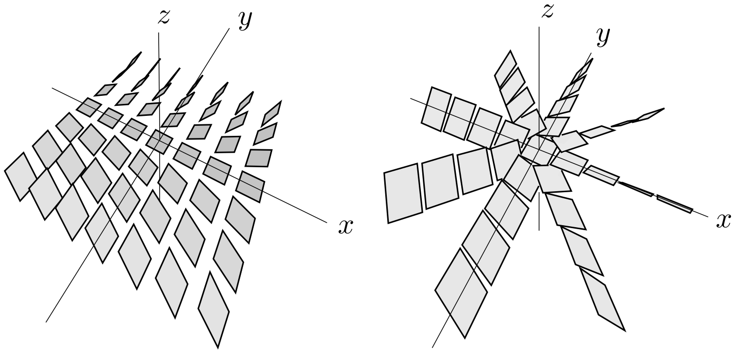


Figure 1: On the left: the standard contact structure $\xi_{st} = \ker(dz - ydx)$. On the right: the isomorphic contact structure $\xi'_{st} = \ker(dz + xdy - ydx)$. (Figures by S. Schönenberger)

Proposition 2.12. *For any point $p \in S^3$ there is a diffeomorphism $\phi\colon S^{2n-1} \setminus \{p\} \to \mathbb{R}^{2n-1}$ such that $d\phi(\xi_{st}) = \xi_{st}$.*

The proof is an elementary, but very tedious computation, which can be found in [13].

The importance of the standard contact structure in $\mathbb{R}^{2n+1}$ relies on the fact that it is the local model for every contact structure.

Theorem 2.13. *(Darboux) Let (M, ξ) be a contact $(2n+1)$-manifold. Any point $p \in M$ has an open neighbourhood U with a diffeomorphism $\phi\colon U \to V \subset \mathbb{R}^{2n+1}$ such that $\phi(p) = 0$ and $\alpha = \phi^*\alpha_{st}$.*

We prove the theorem for $n = 1$. The general case can be treated with similar ideas, but requires more technical sophistication.

Proof. In the proof we will allow ourselves to shrink open neighbourhoods without changing their names.

Fix $p \in M$. By the flow-box theorem we can find a neighbourhood U of p in M and a diffeomorphism $\phi_0\colon U \to V_0 \subset \mathbb{R}^{2n+1}$ such that $\phi_0(p) = 0$ and $d\phi_0(R) = \partial_z$. Then $\alpha = \phi_0^*\alpha_0$, where $\alpha_0 = dz + f(x,y)dx + g(x,y)dy$.

Let Y be a z-invariant vector field defined on a neighbourhood of 0 in V_0 such that $Y_p \in \ker(\alpha_0)_p$ for all $p \in V_0$. Then we can find a neighbourhood V_1 of $0 \in \mathbb{R}^{2n+1}$ and a diffeomorphism $\phi_1\colon V_0 \to V_1$ of the form $\phi_1(x,y,z) = (\psi_1(x,y), \psi_2(x,y), z)$ such that $\phi_1(0) = 0$ and $d\phi(Y) = \partial_y$. Then $\alpha_0 = \phi_1^*\alpha_1$, where

$$\alpha_1 = dz - h(x,y)dx$$

for a function h such that $h(0,0,0) = 0$. The contact condition (1) implies that $\frac{\partial h}{\partial y} > 0$, so the map $\phi_2 \colon (x,y,z) \to (x, h(x,y), z)$ is invertible in a neighbourhood of 0, and $\phi = \phi_2 \circ \phi_1 \circ \phi_0$ is the required diffeomorphism. □

We discuss three more examples of contact manifolds. The first one is a quite general construction which relates contact geometry to symplectic geometry.

Definition 2.14. Let (W, ω) be a symplectic manifold. A *Liouville vector field* for (W, ω) is a (possibly locally defined) vector field Y such that $\mathcal{L}_Y \omega = \omega$. A *contact type hypersurface* $V \subset M$ is a hypersurface which is transverse to a Liouville vector field.

Example 2.15. The 1-form $\alpha = (\iota_Y \omega)|_{TV}$ is a contact form on V.

Exercise 2.16. Prove that α is a contact form and that there is an embedding $i \colon (-\epsilon, \epsilon) \times V \to W$ such that $i(\{0\} \times V) = V$ and $i^* \omega = d(e^s \alpha)$.

Let N be a smooth manifold, and choose a Riemannian metric g on N. Then g induces a scalar product g^* on the fibres of the cotangent bundle T^*M. We define the *unit cotangent bundle* S^*N of N as

$$S^*N = \{(p,l) : p \in N, l \in T^*_p M \text{ and } g^*(l,l) = 1\}.$$

We denote by $\pi \colon S^*N \to N$ the projection $\pi(p,l) = p$.

Example 2.17. The 1-form λ on S^*N defined by

$$\lambda_{(p,l)}(v) = l(d\pi(v))$$

for all $v \in T_{(p,l)}(S^*N)$ is a contact form. If $\xi = \ker \lambda$, then

$$\xi_{(p,l)} = d\pi^{-1}(\ker l).$$

Exercise 2.18. Prove that λ is a contact form by showing that Example 2.17 is an instance of Example 2.15.

Exercise 2.19. Show that the Reeb flow of λ on S^*N corresponds to the geodesic flow of g on the unit tangent bundle SN by the identification $S^*N \cong SN$ induced by g.

Exercise 2.20. Prove that the universal cover of S^*S^2 is diffeomorphic to S^3, and that the pull back to S^3 of the contact structure on S^*S^2 defined in Example 2.17 is the standard contact structure on S^3.

The reader be warned: the two previous exercises are difficult.

Example 2.21. If N is a smooth manifold, we define its 1-jet manifold J_N as $J_N = \mathbb{R} \times T^*N$. If λ_N is the Liouville 1-form on T^*N written as pdq in local coordinates, then $dz - \lambda_N$ is a contact form on J_N.

Note that, if $N = \mathbb{R}^n$, the construction of Example 2.21 yields the standard contact form on $\mathbb{R}^{2n+1}$.

Exercise 2.22. Let $f\colon N \to \mathbb{R}$ be a smooth function. Prove that $p \mapsto (f(p), d_p f)$ parametrises a Legendrian submanifold of J_N.

If $\dim N = n$, then J_N is a real vector bundle of rank $n + 1$. We denote by 0_N its zero section. With the help of jet manifolds we can state a version of Darboux's theorem for Legendrian submanifolds.

Theorem 2.23. *Let (M, ξ) be a contact manifold and $N \subset M$ a Legendrian submanifold. Then for any contact form α of ξ there is a neighbourhood U of 0_N in J_N and an embedding $i\colon U \to M$ such that $i(0_n) = N$ and $i^*\alpha = dz - \lambda_N$.*

Exercise 2.24. Try to prove Theorem 2.23 when $\dim M = 3$ by adapting the proof of Theorem 2.13.

Remark 2.25. Theorem 2.23 together with Exercise 2.22 show that there are "many" Legendrian submanifolds near a given one. This observation will allow us to use some genericity arguments when working with Legendrian submanifolds.

We finish this section by stating two classical theorems in contact topology. We will make no use of them in the following part of this manuscript, but they are important to give the reader the flavour of the subject.

Theorem 2.26. *(Lutz–Martinet) Every oriented plane field in any orientable closed 3-manifold is homotopic to a contact structure.*

In higher dimension there is a topological obstruction to the existence of a contact structure: if a $(2n+1)$-dimensional manifold M admits a contact structure, then its tangent bundle TM admits a reduction of its structure group to $U(n)$. Casals, Pancholi and Presas have recently announced the existence of a contact structure on every 5-manifold admitting such a reduction: see [1]. On the other hand, the problem of determining what manifolds of dimension at least 7 admit a contact structure is still open at the time of writing.

Theorem 2.27. *(Gray) Let ξ_t, $t \in [0, 1]$ be a smooth family of contact structure (i. e. defined by a smooth family of contact forms α_t) on a closed manifold M. Then there is a smooth family of diffeomorphisms $\phi_t\colon M \to M$ such that $\phi_0 = id$ and $d\phi_t(\xi_0) = \xi_t$.*

Be careful that the statement does not say that $\alpha_0 = \phi_t^*\alpha_t$. This is usually not true because the dynamic of the Reeb vector field is very sensitive to the choice of the contact form. Also the theorem is not true on non-compact manifolds.

2.2 Legendrian knots

Definition 2.28. Let L_0 and L_1 be two Legendrian submanifolds in a contact manifold (M, ξ). We say that they are *Legendrian isotopic* if there is a map $f: L \times [0,1] \to M$ such that:

1. $f|_{L \times \{t\}}$ is an embedding for all $t \in [0,1]$,
2. $f(L \times \{t\})$ is Legendrian for all $t \in [0,1]$, and
3. $f(L \times \{t\}) = L_t$ for $t \in \{0,1\}$.

This definition is equivalent to the following, apparently stronger, property.

Lemma 2.29. *(See [13, Theorem 2.6.2]). Two Legendrian submanifolds L_1 and L_2 of a contact manifold (M, ξ) are Legendrian isotopic if and only if there exists a smooth family of diffeomorphisms $\varphi_t: M \to M$ such that:*

1. *φ_0 is the identity,*
2. *$\varphi_1(L_0) = L_1$, and*
3. *$d\varphi_t(\xi) = \xi$.*

In the rest of this notes we focus on Legendrian knots in $\mathbb{R}^3$ equipped with the standard contact structure. The main problem we are going to address is to find computable invariants of Legendrian knots which distinguish between non Legendrian isotopic Legenrian knots. The first obvious invariant is the (topological) *knot type*: if two Legendrian knots are Legendrian isotopic, they are *a fortiori* smoothly isotopic. Another important problem, which we will not consider in these notes, is the classification of the Lagrangian isotopy classes of Legendrian representatives in a given topological knot type.

Knots in $\mathbb{R}^3$ are usually represented by their projection to a plane. For Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ there are two geometrically significant projections:

- the *front projection* $\Pi: (x, y, z) \mapsto (x, z)$, and
- the *Lagrangian projection* $\pi: (x, y, z) \mapsto (x, y)$.

The remarkable property of these two projections is that a Legendrian knot L can be recovered from either $\Pi(L)$ or $\pi(L)$ (in the case of the Lagrangian projection, up to translations in the z-direction).

Let us consider the front projection first. Let $t \mapsto (x(t), y(t), z(t))$ be the parametrisation of a Legendrian arc, then $\dot{z}(t) - y(t)\dot{x}(t) = 0$, so we can recover $y(t)$ by

$$y(t) = \frac{\dot{z}(t)}{\dot{x}(t)}, \tag{4}$$

that is from the slope of the tangent to the front projection.

Now let us turn to the Lagrangian projection. If $t \mapsto (x(t), y(t), z(t))$ is again the parametrisation of a Legendrian arc, then we can recover $z(t)$ from the Lagrangian projection $(x(t), y(t))$ by

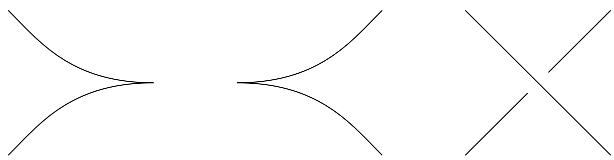
$$z(t) = z(0) + \int_0^t y(s)\dot{x}(s)ds. \tag{5}$$

Observe that we need to know $z(0)$ in order to be able to recover $z(t)$, which means that the Lagrangian projection determines the arc up to translations in the z direction.

In order to make use of the correspondence between Legendrian knots and their projections, we need to characterise the plane curves which can arise as front or Lagrangian projection of a Legendrian knot. We consider the front projection first.

Proposition 2.30. *(See for example [13, Lemma 3.2.3] or [6, Section 2.3].) For a generic Legendrian knot L, the front projection $\Pi(L)$ is embedded outside a finite number transverse points and cubic cusps, and its tangent line (which is well defined even at the cusps) is never vertical.*

If we imagine to look at the knot from $y = -\infty$, at every double point we will see the strand with negative slope above the strand with positive slope, therefore a generic front will have the following singularities:



The relation between Legendrian knots and front projections is an equivalence in the strongest possible sense:

Proposition 2.31. *Every plane curve with isolated cusps and (possibly) transverse double points and with no vertical tangents (including at cusps), is the front projection of a unique Legendrian knot.*

Remark 2.32. Front projections of Legendrian knots are usually drawn with horizontal cusps. This is however only an aesthetic choice; In fact cusps with every slope can and do occur "in nature", with the only exception of vertical cusps.

The first result we prove about Legendrian knots is the following approximation theorem.

Theorem 2.33. *Every arc in a contact 3-manifold (M, ξ) can be C^0 approximated by a Legendrian arc which is smoothly isotopic to the original one relative to the endpoints.*

Proof. By Darboux's Theorem it is enough to prove the statement for arcs in $(\mathbb{R}^3, \xi_{st})$. Let C be an arc in $\mathbb{R}^3$. It is is possible to draw a zig-zag in the xz-plane which is C^0 close to $\Pi(C)$, and whose slope is C^0-close to the y-coordinate of C. Then the zig-zag is the front projection of a Legendrian arc which is C^0-close to C. □

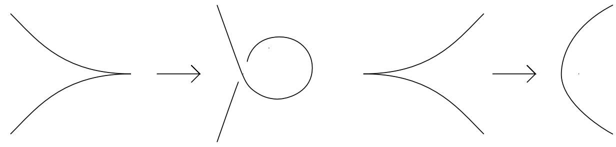
Example 2.34. ("Real life" example) We can model a car as a small arrow, whose state is determined by its position in $\mathbb{R}^2$ and by its orientation, so its configuration space is $\mathbb{R}^2 \times S^1$. We put coordinates (x, y) on $\mathbb{R}^2$, and θ on S^1. The car is not free to move in any direction: it can only go straight or turn, but cannot translate laterally. This means that its allowed velocities belong to the 2-dimensional distribution spanned by the vector fields $\cos(\theta)\partial_x + \sin(\theta)\partial_y$ and ∂_θ. This distribution is a contact structure with contact form $\alpha = \sin(\theta)dx - \cos(\theta)dy$, and the allowed trajectories of the car are Legendrian curves. Then Theorem 2.33 implies that we can park the car, which means that we can manoeuvre the car in order to make it travel arbitrarily close to a forbidden trajectory.

Let us consider now the Lagrangian projection. It is immediate to see that the Lagrangian projection is always an immersion because ∂_z never belongs to ξ_{st}. Therefore, for a generic Legendrian knot L, its Lagrangian projection $\pi(L)$ is an immersed curve with only double points. Moreover, every immersed curve with double point lifts to an *immersed Legendrian arc* which is unique up to translations in the z-direction. The following lemma is an easy consequence of Equation (5) and Stokes's theorem.

Lemma 2.35. *An immersed plane curve is the Lagrangian projection of a Legendrian knot if and only if it encircles a region of signed area zero, but none of its sub-arcs with both ends on the same double point encircles a region of signed area zero.*

The advantage of the front projection is that the conditions which characterise fronts are qualitative and therefore can be easily checked from the diagram. On the other hand, some invariant are more easily defined starting from a Lagrangian projection. Luckily it is not difficult to pass from the front projection to the Lagrangian projection.

Proposition 2.36. *(See [6, Theorem 4.19].) If L is a Legendrian knot, we obtain an immersed curve which is isotopic to the Lagrangian projection of a Legendrian isotopic knot by applying the following local modification to $\Pi(L)$:*



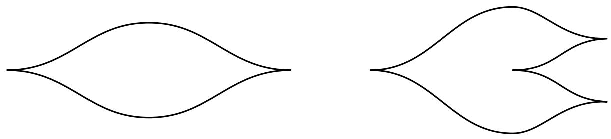


Figure 2: Front projections of Legendrian unknots.

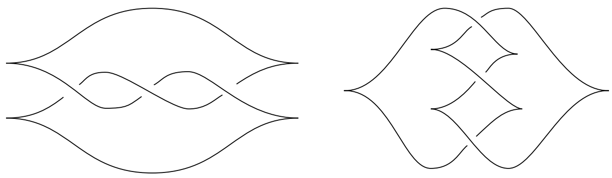


Figure 3: Front projections of a Legendrian right-handed trefoil knot (left) and of a left-handed trefoil knot (right).

Remark 2.37. Before going further in the study of Legendrian knots, we should make clear a subtlety about the Lagrangian projection. As we have seen, an immersed curve must satisfy some area constraints to be the Lagrangian projection of a Legendrian knot. However, the condition of being *isotopic* to a Lagrangian projection is a topological condition. In this course we will define several invariants of Legendrian knot with the aid of the Legendrian projection, but for all of them, the computation will depend only on the isotopy class of the projection. For this reason we will never be careful with the area constraints in our figures.

2.3 Classical invariants

To Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ — and, more generally, to null-homologous Legendrian knots in any contact manifold — we can associate two numerical invariants: the *Thurston-Bennequin number* and the *rotation number*. They are called the *classical* invariants of a Legendrian knot.

Definition 2.38. The *Thurston–Bennequin number* $\mathtt{tb}(L)$ of a Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st})$ is defined as follows. We take a small translate L' of L in the direction of a transverse vector field (for example ∂_z). Then we define

$$\mathtt{tb}(L) = \mathtt{lk}(L, L').$$

The Thurston-Bennequin number measures the difference between the framing induced by ξ with the framing induced by a Seifert surface. The latter enter the definition through the linking number, that can be defined as the algebraic intersection between L' and a Seifert surface of L.

Definition 2.39. The *rotation number* $\mathtt{rot}(L)$ of an oriented Legendrian knot $L \subset (\mathbb{R}^3, \xi_{st})$ is defined as follows. Let Σ be a Seifert surface for L, and let $\tau \colon \xi|_\Sigma \to \mathbb{C}$ be a trivialisation of $\xi|_\Sigma$ (which always exists because Σ retracts on a 1-dimensional cell complex). If $\gamma \colon S^1 \to L$ is an orientation preserving parametrisation of L, the composition $\tau \circ \dot{\gamma}$ gives a map $\tau \circ \dot{\gamma} \colon S^1 \to \mathbb{C}^*$. We define

$$\mathtt{rot}(L) = \deg(\tau \circ \dot{\gamma}).$$

Lemma 2.40. *The rotation number of L does not depend on the Seifert surface and the trivialisation chosen, and it changes sign if we change the orientation of L.*

Proof. The proof is left as an exercise for the reader. □

The following theorem is a real cornerstone of contact topology in dimension three, and marked its coming to age as a subfield of topology.

Theorem 2.41. *(Bennequin) Let L be a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$, and Σ be a Seifert surface of L. Then*

$$\mathtt{tb}(L) + |\mathtt{rot}(L)| \leq -\chi(\Sigma). \tag{6}$$

There are several proofs of the Bennequin inequality. The most notable are Bennequin's original one, which is purely topological, Eliashberg's one, which uses holomorphic curves, and the proof by Lisca and Matić using Seiberg-Witten theory. Unfortunately we will have no time to discuss any of them.

The definitions of the Thurston-Bennequin number and of the rotation number make sense without modifications for null-homologous Legendrian knots in any contact 3-manifold. The only difference is that the rotation number depends on the Seifert surface used to define it if the Euler class of ξ is non-trivial. However the Bennequin inequality (6) is not a general property of contact structures.

Definition 2.42. A contact manifold is called *tight* if the Thurston-Bennequin (6) holds for every null-homologous Legendrian knot. Otherwise it is called *overtwisted*.

Tight contact structures are usually considered the interesting ones, because they reflect topological properties of the underlying manifold. The standard contact structure on $\mathbb{R}^3$ is tight by Theorem 2.41.

The reader may have already met a different definition of an overtwisted contact structure. There are several ones, which are all equivalent. The most common one is that a contact structure is overtwisted if it admits an *overtwisted disc*, which is an embedded disc D such that the contact planes are tangent to D along ∂D. An overtwisted disc violates Bennequin's inequality because $\mathtt{tb}(\partial D) = 0$, and it is a

Figure 4: On the left: a positive crossing. On the right: a negative crossing

surprising (but not very hard) fact proved by Eliashberg that the violation of the Bennequin inequality for *any* knot implies the existence of an overtwisted disc.

Example 2.43. Take cylindrical coordinates (r, θ, z) on $\mathbb{R}^3$. The contact structure ξ_{ot} on $\mathbb{R}^3$ defined by the contact form $\alpha_{ot} = \cos(r)dz + r\sin(r)d\theta$ is overtwisted.

Exercise 2.44. Find the overtwisted disc for $(\mathbb{R}^3, \xi_{ot})$.

The classical invariants for Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ can be easily computed from their front and Lagrangian projection. Given an immersion $\gamma\colon S^1 \to \mathbb{R}^2 \cong \mathbb{C}$, we define the *winding number* of γ as

$$\mathtt{wind}(\gamma) = \mathtt{deg}(\dot{\gamma}).$$

Given a knot diagram $\pi(K)$, we call the *write* of $\pi(K)$ the algebraic count of its crossings, where the sign is chosen as described in Figure 4.

Proposition 2.45. *If L is a Legendrian knot in (R^3, ξ_{st}), then*

$$\mathtt{tb}(L) = \mathtt{writhe}(\pi(L)) \quad \textit{and} \quad \mathtt{rot}(L) = \mathtt{wind}(\pi(L)).$$

Exercise 2.46. Derive formulas to compute the classical invariants of a Legendrian knot from its front projection.

Definition 2.47. A smooth knot type is called *Legendrian simple* if its Legendrian representatives are classified, up to Legendrian isotopy, by their Thurston–Bennequin and rotation numbers.

The following knots types are Legendrian simple:

- unknot [5]
- torus knots [8]
- figure-eight [8]

and a few others: [9, 11]. However, not all knot types are Legendrian simple:

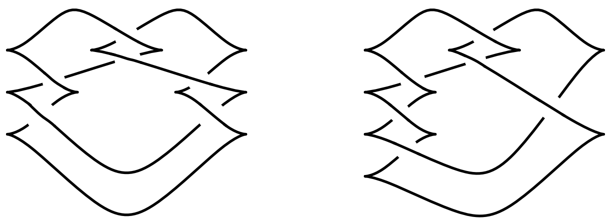


Figure 5: Chekanov's examples in front projection. (Figure by Vera Vértesi)

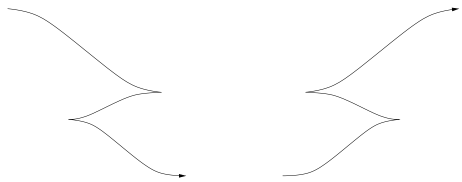


Figure 6: Stabilisations in front projection: an upward zig-zag on the left, and a downward zig-zag on the right.

Theorem 2.48. *(Chekanov, [2]) The mirror of the knot 5_2 has two non isotopic Legendrian representatives with the same Thurston–Bennequin and rotation numbers; see Figure 5.*

Note however that the knot type 5_2 il Legendrian simple [11]. This shows that contact topology is very sensitive to chirality.

In order to distinguish his examples, Chekanov introduced a new invariant for Legendrian knots, which is called either *Legendrian contact homology* or *Chekanov's dga*. We will describe this invariant in the next section, and will develop the theory enough that distinguishing the two Legendrian knots in Figure 5 will become an exercise.

2.4 Stabilisation

We finish this section by describing an operation called *stabilisation* which transforms a Legendrian knot into another Legendrian knot which is smoothly isotopic and has smaller Thurston–Bennequin number. For Legendrian knots in $(\mathbb{R}^3, \xi_{st})$ stabilisation is performed by adding a zig-zag to their front projection, and its definition can be extended to Legendrian knots in any contact manifold by performing it inside a Darboux ball. If the zig-zag is downward we say that the stabilisation is positive, and if it is upward, we say that the stabilisation is negative: see Figure 6. We denote the positive stabilisation by S_+ and the negative one by S_-. It is possible to check that the result of a stabilisation does not depend, up to Legendrian isotopy, on where the zig-zag is added.

The effect of the S_+ and S_- on the classical invariants can be easily computed from the Lagrangian projection:

Lemma 2.49. $\mathtt{tb}(S_\pm(L)) = \mathtt{tb}(L) - 1$ *and* $\mathtt{rot}(S_\pm(L)) = \mathtt{rot}(L) \pm 1$.

Stabilisations add some flexibility to Legendrian knots:

Theorem 2.50. *(See [12].) Any two smoothly isotopic Legendrian knots become Legendrian isotopic after a finite number of stabilisations.*

3 Legendrian contact homology

3.1 Differential graded algebras

In this section we introduce *differential graded algebras*, which are algebraic objects we will use in the study of Legendrian knots. We fix a commutative ring R, which in the applications will be $\mathbb{Z}/2\mathbb{Z}$.

Definition 3.1. An R*-algebra* $\mathcal{A}$ is an R-module[5] with a bilinear map (multiplication)

$$m\colon \mathcal{A} \otimes_R \mathcal{A} \to \mathcal{A}$$

which is associative, i.e $m(x \otimes m(y \otimes z)) = m(m(x \otimes y) \otimes z)$ for all $x, y, z \in \mathcal{A}$.

In these notes we will not assume that the product is commutative. For simplicity we will use the notation $x \cdot y$ to denote $m(x \otimes y)$, and we will suppress the ring R from the notation when it is clear from the context.

Definition 3.2. Let G be a cyclic group, and $\mathcal{A}$ an R-algebra. We say that $\mathcal{A}$ is G*-graded* (or, more simply, graded) if $\mathcal{A}$ as an R-module decomposes as a direct sum

$$\mathcal{A} = \bigoplus_{n \in G} \mathcal{A}^n$$

and, for any $x \in \mathcal{A}^n$ and $y \in \mathcal{A}^m$, we have $m(x \otimes y) \in \mathcal{A}^{n+m}$.

If x is an omogeneous element of degree n (i.e. $x \in \mathcal{A}^n$) we will write $|x| = n$. The definitions one can find in the literature usually consider only $\mathbb{Z}$-gradings. However in Legendrian contact homology we will need also G-gradings when G is a finite cyclic group.

Definition 3.3. A *differential graded algebra* (dga) $(\mathcal{A}, \partial)$ is a graded algebra $\mathcal{A}$ with an R-linear map $\partial\colon \mathcal{A} \to \mathcal{A}$ (called differential) such that:

[5]If you are not familiar with modules over a ring, consider a vector space over a field

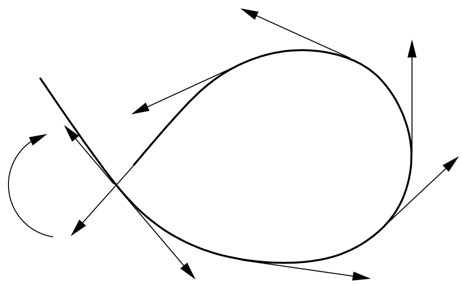


Figure 7: Computing the grading of a double point

1. $\partial \circ \partial = 0$ (in particular $\mathcal{A}$ is a chain complex),
2. $\partial(\mathcal{A}^n) \subset \mathcal{A}^{n-1}$ (i.e. ∂ has degree -1)[6], and
3. the differential satisfies the (graded) Leibnitz rule: if $x \in \mathcal{A}^n$ and $y \in \mathcal{A}^m$, then $\partial(x \cdot y) = (\partial x) \cdot y + (-1)^n x \cdot (\partial y)$.

3.2 Chekanov's dga

Given the Lagrangian projection of a Legendrian knot, we are going to associate a differential graded algebra (dga) to it, whose homology will be an invariant of the knot under Legendrian isotopies. The construction of this algebra is due to Chekanov [2] and is motivated by the (conjectured) construction of Legendrian contact homology by Eliashberg, Givental, and Hofer [4].

The algebra. Given a generic Legendrian knot L, we denote by $\mathcal{C}$ the finite set of the double points of its Lagrangian projection $\pi(L)$. Let $\mathcal{A}_L$ be the free non-commutative algebra over $\mathbb{Z}/2\mathbb{Z}$ generated by $\mathcal{C}$. The algebra $\mathcal{A}_L$ is unital, and 1 corresponds to the empty word.

The grading. We define a function $\mathcal{C} \to \mathbb{Z}/(2\,\mathtt{rot}(L))$ as follows. Given a double point $a \in \mathcal{C}$, let $\gamma \colon [0, \pi] \to \pi(L)$ be a regular path from a to itself, starting from the upper strand (the one with bigger z-coordinate) and arriving to the lower strand (the one with smaller z-coordinate). Then define $\Gamma \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{RP}^1$ by taking the projection of $\dot\gamma(t)$ for $t \in [0, \pi]$ and the clockwise rotation from $[\dot\gamma(\pi)]$ to $[\dot\gamma(0)]$ for $t \in [\pi, 2\pi]$; see Figure 3.2. Then we define

$$|a| = \mathtt{deg}(\Gamma) \pmod{2\,\mathtt{rot}(L)}$$

and we extend this function to a grading on $\mathcal{A}_L$ by requiring that $|ab| = |a| \cdot |b|$. The function $\mathtt{deg}(\Gamma)$ depends on the choice of the path, but it is easy to see, from Proposition 2.45, that different choices of paths give results which differ by a multiple of $2\,\mathtt{rot}(L)$. The reason

[6] Differential graded algebras where the differential has degree $+1$ are also common.

Figure 8: On the left: the "up" quadrants. On the right: the "down" quadrants.

of the factor 2 is that we have projectivised $\mathbb{R}^2 \setminus \{0\}$ before computing the degree.

The differential. We define the differential for a single double point first, and extend it to $\mathcal{A}_L$ by the Leibniz formula $\partial(b_1 b_2) = (\partial b_1)b_2 + (-1)^{|b_1|}b_1(\partial b_2)$.[7] The two strands of $\pi(L)$ at a double point locally divide the plane into four quadrants. Two of them are "*up*" quadrants, and the other two are "*down*" quadrants: see Figure 8. If we orient the boundary of a quadrant with the orientation induced by the usual orientation of $\mathbb{R}^2$, the quadrant is "up" if the lower strand comes before the upper strand, and is "down" if the contrary happens.

Given $b, a_1, \dots, a_n \in \mathcal{C}$, we define $\Delta(b; a_1, \dots, a_n)$ as the set of immersed polygons in $\mathbb{R}^2$ with edges on $\pi(L)$, vertices at $b, a_1, \dots a_n$, and which cover an "up" quadrant near b and "down" quadrants near $a_1, \dots, a_n$. We denote by $\#\Delta(b; a_1, \dots, a_n)$ the number of elements in $\Delta(b; a_1, \dots, a_n)$, reduced mod 2. Then we define the differential by:

$$\partial b = \sum_{n \in \mathbb{N}} \sum_{(a_1, \dots, a_n) \in \mathcal{C}^n} \#\Delta(b; a_1, \dots, a_n) a_1 \dots a_n.$$

The differential makes sense because each $\Delta(b; a_1, \dots, a_n)$ is a finite set, and the sum is finite because of the following area considerations. For every double point $c \in \mathcal{C}$, let c_+ be its preimage in L with bigger z-coordinate, and c_- be the one with smaller z-coordinate. Denote by $z(c_\pm)$ the z-coordinate of $c_\pm$, and define $h(c) = z(c_+) - z(c_-)$. Then $h(c) > 0$ for every double point $c \in \mathcal{C}$.

Lemma 3.4. *If* $\Delta(b; a_1, \dots, a_n) \neq \emptyset$*, then*

$$h(b) > \sum_{i=1}^{n} h(a_i). \tag{7}$$

Proof. Let $P \in \Delta(b; a_1, \dots, a_n)$ be a polygon, and $\widetilde{\partial P}$ be the lift of ∂P to a disconnected Legendrian path contained in L such that $\pi(\widetilde{\partial P}) =$

[7] Of course the sign is superfluous in the theory over $\mathbb{Z}/2\mathbb{Z}$ discussed here.

∂P. Then

$$h(b) - \sum_{i=1}^{n} h(a_i) + \int_{\widetilde{\partial P}} dz = 0.$$

From the relation $dz = ydx$ on $\widetilde{\partial P}$ and Stokes theorem we obtain

$$h(b) - \sum_{i=1}^{n} h(a_i) \geq \int_P dx \wedge dy > 0.$$

□

Given any $b \in \mathcal{C}$ there is only a finite number of words $(a_1, \dots, a_n)$ such that the inequality (7) holds, therefore Lemma 3.4 implies that the sum in the differential is a finite sum.

Theorem 3.5. *(Chekanov, [2]) ∂ has degree -1, $\partial^2 = 0$ and the homology $H_*(\mathcal{A}_L, \partial)$ is an invariant of L by Legendrian isotopies.*

We will not prove Theorem 3.5. ∂ has degree -1 as a consequence of the definition of the degree in $\mathcal{A}_L$ and of the well known fact that the sum of the exterior angles of a polygon is always π. The idea of the proof that $\partial^2 = 0$ is that ∂^2 counts pairs of immersed polygons with a common vertex, which is "down" for the first disc, and "up" for the second one. The union of these two polygons is a region with an obtuse angle which can be decomposed in two different ways, so the contributions to $\partial^2 = 0$ come in pairs which cancel each other: see Figure 9.

In order to prove invariance, one should introduce a Legendrian analogue to the Reidemeister's moves and keep track of how the complex changes when performing a move.

Exercise 3.6. Prove that ∂ has degree -1.

Definition 3.7. The homology of $(\mathcal{A}_L, \partial)$ is called *Legendrian contact homology* of L and is denoted by $CH_*(L)$.

3.3 Computations and examples

In order to illustrate the definition of the Chekanov's dga, we compute it in the simplest non-trivial case.

Example 3.8. Let L be a Legendrian right-handed trefoil knot with $\mathtt{tb}(L) = 1$ and $\mathtt{rot}(L) = 0$. The algebra $\mathcal{A}_L$ is generated by five intersection points a_1, a_2, b_1, b_2, b_3 of degrees $|a_i| = 1$, $|b_i| = 0$. The differential is:

- $\partial a_1 = 1 + b_1 + b_3 + b_3 b_2 b_1$
- $\partial a_2 = 1 + b_1 + b_3 + b_1 b_2 b_3$

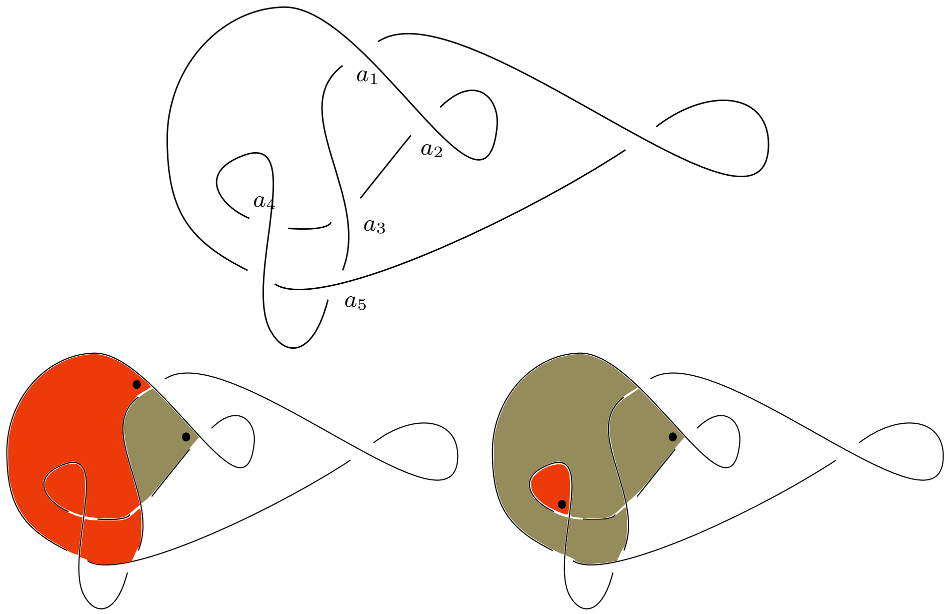


Figure 9: A pair of cancelling contributions to ∂^2.

- $\partial b_1 = \partial b_2 = \partial b_3 = 0$.

The computation of $\mathcal{A}_L$ is explained in Figure 10, where the shaded regions represent the discs contributing to the differential of a_1.

Legendrian contact homology is a very powerful invariant, but it has a disappointing weaknesses: it vanishes for a very big class of Legendrian knots.

Theorem 3.9. *(Chekanov [2]) If L is obtained by stabilising some other Legendrian knot, then $CH_*(L) = 0$.*

Proof. If L is a stabilisation, then up to isotopy $\pi(L)$ contains a loop as in Figure 11. Moreover this loop can be made arbitrarily small by a Legendrian isotopy of L. This is a property of stabilised knots: in fact every Lagrangian projection contains loops, but in general they are big.

Let c be the double point associated to the small loop. The shaded region Figure 11 is a monogone with its up vertex at c. This implies that $\partial c = 1 + \ldots$. By Stokes theorem $h(c)$ is equal to the area of the loop, so it can be made as small as we want. If we make it smaller than $h(a)$ for any other double point a of $\pi(L)$, then only monogones can contribute to ∂c by Lemma 3.4. However, if c is the up vertex of a second monogone, then L is an unstabilised unknot. Therefore, if L is

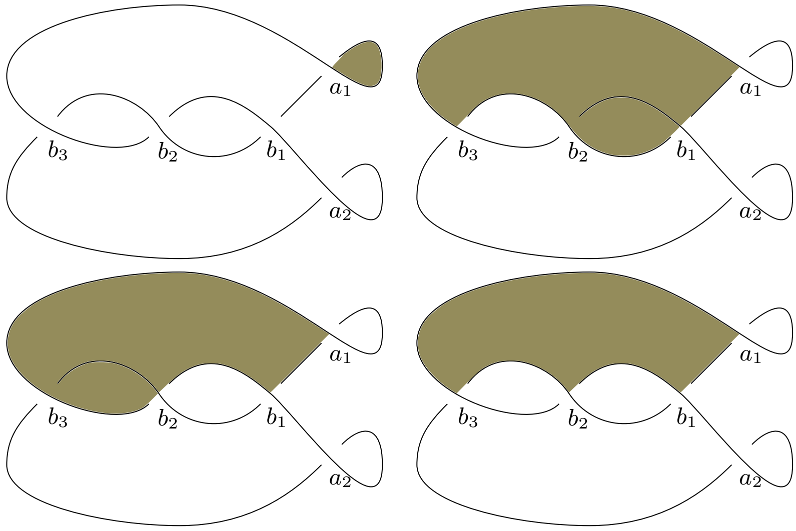


Figure 10: Chekanov's algebra differential for the trefoil knot

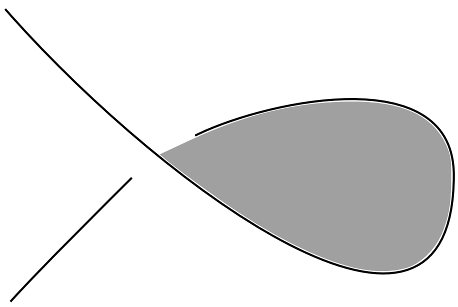


Figure 11: The little loop and its monogone in a stabilised knot.

a stabilised knot, the monogone in Figure 11 is the only contribution of to ∂c, so $\partial c = 1$. This implies $HC_*(L) = 0$: in fact, if $w \in \mathcal{A}_L$ is a cycle, then by the Leibniz formula we have $\partial(cw) = (\partial c)w - c(\partial w) = w$. □

3.4 Augmentations

Even if Chekanov's dga is described combinatorially, Legendrian contact homology is very hard to compute because of the non-Abelian and non-linear nature of the differential. However, in some lucky situations, we can simplify the complex by an algebraic operation which we are going to describe. We introduce the *word-length filtration* on $\mathcal{A}_L$: Let $\mathcal{A}_L^n \subset \mathcal{A}_L$ be the subalgebra generated, as a vector space, by all words in $\mathcal{C}$ of length bigger or equal than n. It is easy to see that every $\mathcal{A}_L^n$ is not only a subalgebra, but also a bilateral ideal. Moreover they define an infinite descending filtration

$$\mathcal{A}_L = \mathcal{A}_L^0 \supset \mathcal{A}_L^1 \supset \ldots \supset \mathcal{A}_L^n \supset \ldots$$

Unfortunately the differential does not preserve the word length filtration: it would if and only if ∂c had no constant term for every $c \in \mathcal{C}$, and one can see that this is never the case. However we often can conjugate the differential by an automorphism of $\mathcal{A}_L$ in order to kill the unwanted constant terms.

Definition 3.10. An *augmentation* of $\mathcal{A}_L$ is an algebra homomorphism

$$\epsilon \colon \mathcal{A}_L \to \mathbb{Z}/2\mathbb{Z}$$

such that:

1. $\epsilon(1) = 1$,
2. $\epsilon \circ \partial = 0$,
3. $\epsilon(a) = 0$ if $|a| \neq 0$.

Not every Legendrian knot admits an augmentation: in fact the existence of an augmentation implies that $HC_*(L) \neq 0$, so in particular stabilised Legendrian knots admit no augmentation.

Remark 3.11. Augmentations for a given Legendrian knot L can be determined explicitly: if we denote by C_0 the vector space over $\mathbb{Z}/2\mathbb{Z}$ generated by the intersection points in $\mathcal{C}$ with degree 0, then the condition $\epsilon \circ \partial = 0$ is a polynomial system on the dual space C_0^*. The solutions of this system are in one-to-one correspondence with the augmentations of $\mathcal{A}_L$.

Given an augmentation ϵ, we can define an automorphism Φ^ϵ of $\mathcal{A}_L$ by

$$\Phi^\epsilon(a) = a + \epsilon(a)$$

for all $a \in \mathcal{C}$, and we define a new boundary ∂^ϵ by

$$\partial^\epsilon = \Phi^\epsilon \circ \partial \circ (\Phi^\epsilon)^{-1}.$$

Lemma 3.12. *If ϵ is an augmentation, then ∂^ϵ preserves the word length filtration.*

Proof. For simplicity we consider only $\mathcal{A}_L^1$. If $\partial b = c_0 + \sum a_1 \dots a_n$ with $c_0 \in \mathbb{Z}/2\mathbb{Z}$, then

$$\begin{aligned}\partial^\epsilon(b) &= \Phi^\epsilon(\partial(b - \epsilon(b)) = \Phi^\epsilon(c_0 + \sum a_1 \dots a_n) \\ &= c_0 + \sum (a_1 + \epsilon(a_1)) \dots (a_n + \epsilon(a_n)) \\ &= c_0 + \sum \epsilon(a_1) \dots \epsilon(a_n) + \text{terms in } \mathcal{A}^1(L).\end{aligned}$$

However $c_0 + \epsilon(a_1) \dots \epsilon(a_n) = \epsilon(\partial b) = 0$, so $\partial^\epsilon(\mathcal{A}_L^1) \subset \mathcal{A}_L^1$. □

Definition 3.13. Given an augmentation ϵ, we define the *order n Legendrian contact homology*

$$L_n^\epsilon CH_*(L) = H_* \left(\mathcal{A}_L^n / \mathcal{A}_L^{n+1}, \partial^\epsilon \right).$$

If $n = 1$, we call it also *linearised Legendrian contact homology*, and write $L^\epsilon CH_*(L)$.

Theorem 3.14. *(Chekanov [2]) For every $n \geq 1$ the set*

$$\{\textit{Isomorphisms classes of } L_n^\epsilon CH_*(L) : \epsilon \textit{ is an augmentation of } \mathcal{A}_L\}$$

is an invariant of L up to Legendrian isotopies.

Now we describe an equivalent, but more explicit, way to define linearised Legendrian contact homology.

Definition 3.15. We define a complex

$$C = \bigoplus_{c \in \mathcal{C}} \mathbb{Z}/2c$$

with differential

$$\partial^\epsilon b = \sum_{n \in \mathbb{N}} \sum_{(a_1, \dots, a_n) \in \mathcal{C}^n} \sum_{i=1}^{n} \#\Delta(b; a_1, \dots, a_n) \epsilon(a_1) \dots \epsilon(a_{i-1}) \epsilon(a_{i+1}) \dots \epsilon(a_n) a_i.$$

Then $L^\epsilon HC(L) = H_*(C, \partial^\epsilon)$.

The linearised Legendrian contact homology detects the Thurston-Bennequin number. In fact:

Proposition 3.16. *For every augmentation ϵ we have*

$$\chi(L^\epsilon HC_*(L)) = \texttt{tb}(L).$$

Proof. From the definition of the degree, we can see that, for all $c \in \mathcal{C}$, the sign of c as a crossing of $\pi(L)$ is equal to $(-1)^{|c|}$. Then $\chi(L^\epsilon HC_*(L))$ is equal to the writhe of $\pi(L)$ and Proposition 2.45 implies the equality. □

As an example, we compute explicitly the augmentations for the right-handed trefoil knot.

Example 3.17. Let L be the trefoil knot in Figure 10. The reader should refer to Example 3.8 for the computation of $\mathcal{A}_L$. Suppose ϵ is and augmentation for L. By degree reasons $\epsilon(a_i) = 0$. Moreover $\epsilon \circ \partial = 0$ gives

$$1 + \epsilon(b_1) + \epsilon(b_3) + \epsilon(b_1)\epsilon(b_2)\epsilon(b_3) = 0.$$

One can easily see that this equation has the following five solutions:

1. $\epsilon(b_1) = 1, \epsilon(b_2) = 1, \epsilon(b_3) = 1,$
2. $\epsilon(b_1) = 1, \epsilon(b_2) = 0, \epsilon(b_3) = 1,$
3. $\epsilon(b_1) = 1, \epsilon(b_2) = 0, \epsilon(b_3) = 0,$
4. $\epsilon(b_1) = 0, \epsilon(b_2) = 1, \epsilon(b_3) = 1,$
5. $\epsilon(b_1) = 0, \epsilon(b_2) = 0, \epsilon(b_3) = 1.$

Exercise 3.18. Prove that the two Legendrian knots in Figure 5 are not Legendrian isotopic.

Hint. Use Proposition 2.36 to produce Lagrangian projections. Then compute their Chekanov d.g.a.'s. Determine their augmentations and compute their linearised contact homology for all augmentations. You will see that the two sets of homologies are different.

3.5 Geometric motivation

In this section we will describe the geometric motivation behind Chekanov's combinatorial definition. For more details see [4, 7, 10].

Consider the symplectic manifold $(\mathbb{R} \times \mathbb{R}^3, d(e^s\alpha_0))$, which is called the *symplectisation* of the standard contact structure on $\mathbb{R}^3$. In coordinates (s, x, y, z), where s is the coordinate in the first $\mathbb{R}$ factor, the symplectic form is $d(e^s\alpha_0) = e^s(ds \wedge dz - yds \wedge dx + dx \wedge dy)$. On $\mathbb{R} \times \mathbb{R}^3$ we consider also the following almost complex structure J:

1. $J(\partial_s) = \partial_z$
2. $J(\partial_z) = -\partial_s$

3. $J(\partial_x) = \partial_y + y\partial_s$
4. $J(\partial_y) = -\partial_x - y\partial_z$.

The surface $\widehat{L} = \mathbb{R} \times L \subset \mathbb{R} \times \mathbb{R}^3$ is *Lagrangian*, i. e. $d(e^s\alpha_0)|_{\widehat{L}} = 0$. If c is a double point of $\pi(L)$, we call c_+ and c_- the two points in L such that $\pi(c_\pm) = c$ so that c_+ is the point with the highest z-coordinate of the two. Let γ_c be the vertical segment from c_- to c_+.

Remark 3.19. The segment γ_c is called a *Reeb chord* of L, because ∂_z is the Reeb vector field of the contact form α_0, i. e. the vector field R characterised by $\iota_R d\alpha_0 = 0$ and $\alpha_0(R) = 1$.

We will define a new differential ∂_J on $\mathcal{A}_L$ by counting certain J-holomorphic curves in $(\mathbb{R} \times \mathbb{R}^3, J)$. We start by defining the relevant moduli spaces. Let $\mathbb{D}$ be the closed unit disc and $\{x_1, \dots, x_n, y\} \subset \partial\mathbb{D}$ cyclicly ordered points which are not fixed. Then let us consider the set of smooth maps

$$\tilde{u} = (a, u) \colon \mathbb{D} \setminus \{x_1, \dots, x_n, y\} \to \mathbb{R} \times \mathbb{R}^3$$

satisfying:

1. $J \circ d\tilde{u} = d\tilde{u} \circ i$,
2. $\tilde{u}(\partial(\mathbb{D} \setminus \{x_1, \dots, x_n, y\}) \subset \widehat{L}$,
3. $\lim_{z \to y} a(z) = +\infty$ and $\lim_{z \to x_i} a(z) = -\infty$,
4. $\lim_{z \to y} u(z) = \gamma_b$ and $\lim_{z \to x_i} u(z) = \gamma_{a_i}$ (in a suitable sense).

Observe that, if (a, u) is J-holomorphic, then $(a + \kappa, u)$ is also J-holomorphic for every constant $\kappa \in \mathbb{R}$. Let $\mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n})$ be the *moduli space* of J-holomorphic maps $\tilde{u}$ satisfying the conditions above, modulo conformal reparametrisations of the disc, and translations of a by a constant.

One can prove that $\mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n})$ is a smooth manifold of dimension

$$\dim \mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n}) = |b| - \sum_{i=1}^{n} |a_i| - 1$$

which can be compactified by adding strata corresponding to lower dimensional moduli spaces. In particular, moduli spaces of negative dimension are empty, and 0-dimensional moduli spaces are compact.

We define a boundary operator ∂_J on $\mathcal{A}_L$ as follows:

$$\partial b = \sum_{\substack{a_1, \dots, a_n \\ |b| - |a_1| - \dots - |a_n| = 1}} \#\mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n}) a_1 \dots a_n,$$

and we extend it to $\mathcal{A}_L$ by linearity and the Leibniz rule.

Theorem 3.20. *(See [7].)* $\partial_J^2 = 0$ *and* $H_*(\mathcal{A}_L, \partial_J)$ *is an invariant of* L *under Legendrian isotopies.*

Theorem 3.21. *The identity* $(\mathcal{A}_L, \partial) \to (\mathcal{A}_L, \partial_J)$ *is a chain map.*

Proof. The theorem will follow from the fact that the Lagrangian projection induces an identification

$$\pi_* \colon \mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n}) \to \Delta(b; a_1, \dots, a_n)$$

defined as $\pi_*(\tilde{u}) = \pi \circ u$ when $|b| - \sum\limits_{i}^{n} |a_n| = 1$.

From the properties of J it follows that the composite map $\pi \circ u$ is holomorphic and, of course, the boundary of $\mathbb{D}$ is mapped to $\pi(L)$. If $|b| - \sum\limits_{i=1}^{n} |a_n| = 1$ the winding number of the boundary of the image of $\pi \circ u$ is one, so a count of branching degrees implies that $\pi \circ u$ is an immersion. It is easy to see that $\pi \circ u$ covers an "up" quadrant of b in a neighbourhood of y, and "down" quadrants of a_i in a neighbourhood of x_i, and therefore the image of $\pi \circ u$ is an element of $\Delta(b; a_1, \dots, a_n)$.

On the other hand, given an immersed polygon $P \in \Delta(b; a_1, \dots, a_n)$, by the Riemann mapping theorem we can find a holomorphic map $v \colon \mathbb{D} \to P$. We want to lift this map to a J-holomorphic map $\tilde{u} \colon \mathbb{D} \setminus \{y, x_1, \dots, x_n\} \to \mathbb{R} \times \mathbb{R}^3$ of the form $\tilde{u} = (a, v, b)$, where b will be the component of $\tilde{u}$ in the z-coordinate. In coordinates, the Cauchy–Riemann equation $J \circ d\tilde{u} = d\tilde{u} \circ i$ become:

$$\begin{aligned}
da \circ i &= db + v_2 dv_1 \\
db \circ i &= da - v_2 dv_2 \\
dv_1 \circ i &= -dv_2 \\
dv_2 \circ i &= dv_1
\end{aligned}$$

From this it follows that $\triangle b = d(db \circ i) = 0$, so b is harmonic. Moreover the value of b on $\partial \mathbb{D} \setminus \{y, x_1, \dots, x_n\}$ is determined by the fact that (v, b) maps $\partial \mathbb{D} \setminus \{y, x_1, \dots, x_n\}$ on the portion of L which projects on ∂P. We can therefore recover b by solving the Dirichlet problem, and a by integrating the closed form

$$da = -db \circ i - v_2 dv_1 \circ i = -db \circ i + v_2 dv_2.$$

This completes the identification between $\mathcal{M}(\gamma_b; \gamma_{a_1}, \dots, \gamma_{a_n})$ and $\Delta(b; a_1, \dots, a_n)$. □

Augmentations have a geometric counterpart too.

Definition 3.22. If L is a Legendrian knot in $(\mathbb{R}^3, \xi_{st})$, we say that $\Lambda \in \mathbb{R} \times \mathbb{R}^3$ is a *exact Lagrangian filling* of L if

1. $e^s\alpha_{st}|_\Lambda$ is an exact 1-form, and
2. $\Lambda \cap \mathbb{R}^+ \times \mathbb{R}^3 = \mathbb{R}^+ \times L$.

Definition 3.23. We say that a Lagrangian submanifold $\Lambda \subset \mathbb{R}^4$ has Maslov number zero if every disc $u\colon (D^2, \partial D^2) \to (\mathbb{R}^4, \Lambda)$ has Maslov index zero[8].

Proposition 3.24. *An exact Lagrangian filling Λ with Maslov class zero of a Legendrian knot L induces an augmentation ϵ_L of the contact topology algebra $\mathcal{A}_L$.*

Proof. Let $x \in \partial\mathbb{D}$ and let γ_a be a Reeb chord of L. Then we define $\mathcal{M}(\gamma_a)$ to be the moduli space of J-holomorphic maps $\tilde{u} = (a, u)\colon \mathbb{D} \setminus \{x\} \to \mathbb{R} \times \mathbb{R}^3$ such that:

1. $\tilde{u}(\partial\mathbb{D}) \subset \Lambda$,
2. $\lim_{z\to x} a(z) = +\infty$, and
3. $\lim_{z\to x} u(z) = \gamma_a$ (in a suitable sense).

Then $\mathcal{M}(\gamma_a)$ is a smooth manifold[9] of dimension $\dim \mathcal{M}(\gamma_a) = |\gamma_a|$. If $|\gamma_a| = 0$ we define

$$\epsilon_\Lambda(\gamma_a) = \#\mathcal{M}(\gamma_a).$$

□

There is a somewhat surprising relationship between the linearised contact homology $L^{\epsilon_\Lambda}HC(L)$ and the topology of Λ:

Theorem 3.25. *(See [3]) If Λ is an exact Lagrangian filling of L with Maslov number zero and ϵ_Λ is the corresponding augmentation, then*

$$L^{\epsilon_\Lambda}HC(L) \cong H^*(L, \mathbb{Z}/2)$$

up to a degree shift.

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[8] See Mihai Damian's course.

[9] We have to make a slight compactly supported perturbation to J in order to achieve transversality.

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